The Riemann Method in Stochastic Integration

A Brownian motion $X = (X_t)$ $(0 < t,\, X_0 = 0)$ with drift rate μ_t and variance rate σ_t can be constructed from standard Brownian motion $W = (W_t)$ by means of the calculation

$$X_t = \int_0^t \sigma_s dW_s + \int_0^t \mu_s ds.$$

The second integral can be defined as a Lebesgue integral, but the first integral (called a stochastic integral) cannot be so defined, because "almost all" of the sample paths $x(s)$ $(0 < s \leq t,\, x(0) = 0)$ of Brownian motion have unbounded variation. This means that, for any such x, if $\mu_x(\,]u, v])$ is defined as $x(v) - x(u)$, the Lebesgue integral $\int_0^t d\mu_x$ does not exist. This is because the definition of the Lebesgue integral requires us to consider separately those intervals $]u, v]$ for which $x(v) - x(u) \geq 0$ and those for which $x(v) - x(u) < 0$. Considered separately, sums of these quantities diverge to ∞ and $-\infty$ respectively. However, if the generalized Riemann method of Henstock-Kurzweil is used to define the integral there is no such separation of positive and negative values, and for every partition

$$0 = u_0 < u_1 < \cdots < u_{n-1} < u_n = t,$$

of $]0, t]$ the corresponding Riemann sum $\sum \mu_x(\,]u_j, u_{j-1}])$ is

$$(x(u_1) - x(u_0)) + (x(u_2) - x(u_1)) + \cdots + (x(u_n) - x(u_{n-1}))\,,$$

and cancelation gives

$$\sum \mu_x(\,]u_j, u_{j-1}]) = x(u_n) - x(u_0) = x(t).$$

Thus it turns out that $\int_0^t d\mu_x$ exists as a Henstock-Kurzweil integral, with value $x(t)$. Interpreting this in terms of Brownian motion, the stochastic integral $\int_0^t dW_s$ can be defined by partitions and Riemann sums on the interval $]0, t]$, giving

$$\int_0^t dW_s = W_t.$$

Integrals such as $\int_0^t \sigma_s dW_s$ can be similarly defined, and their existence, calculus and values can be easily established under broad conditions. This simplifies the theory and practice of stochastic integration.

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