Dedekind Cuts

Füsun Akman

Calculus was collected into a coherent set of concepts and rules by Leibniz (1646-1716) and Newton (1642-1727), who are regarded as the co-founders of this discipline. Unfortunately, and surprisingly, the foundations were not rigorous at all at first (limits would be defined properly only in the 19th century) but the glory of the applications helped mask this unpalatable fact. One reason why rigor was not possible at the time was that real numbers were not very well understood.

Note that once the existence of integers is assumed[1], it is easy to make up rational numbers by taking pairs of integers (a, b) $(b \neq 0)$ -this corresponds to the fraction a/b- and defining equality (redundancy) by

$$(a, b) = (c, d) \Leftrightarrow ad = bc,$$

addition by

$$(a, b) + (c, d) = (ad + bc, bd),$$

and multiplication by

$$(a, b)\,(c, d) = (ac, bd).$$

Similarly, if you are comfortable with the real numbers, then complex numbers can be defined as pairs (a, b) of real numbers -corresponding to the usual $a + b\,\mathbf{i}$- on which addition and multiplication are defined by

$$(a, b) + (c, d) = (a + c, b + d) \text{ and } (a, b)\,(c, d) = (ac - bd, ad + bc).$$

Exercise: Show that both constructions lead to fields already containing the set you started from.

The above constructions are easy to do and understand because passing from $\mathbf{Z}$ to $\mathbf{Q}$ or from $\mathbf{R}$ to $\mathbf{C}$ does not even require a change in cardinality (the first two are countable, the last two are uncountable). Not so from $\mathbf{Q}$ to $\mathbf{R}$! Clearly, taking pairs of rational numbers is not going to work by because of the cardinalities. Richard Dedekind (1831-1916) was among the first mathematicians who tried to construct reals from rationals without *a priori* assuming their existence. (This endeavor was in general called the *arithmetization of analysis.*) Dedekind was able to build a set of new objects (in fact, pairs!) out of the rationals, which essentially form the real number field.

[1]Kronecker (1823-1891) said: "God made the integers; all else is the work of man."

Definition. A pair (A, B) of nonempty subsets of $\mathbf{Q}$ is called a **(Dedekind) cut** if

1. $A \cup B = \mathbf{Q}$;
2. $A \cap B = \emptyset$;
3. If $a \in A$ and $b \in B$, then $a < b$;
4. The set B has no least element with respect to the ordering of $\mathbf{Q}$.

That is, we understand these two sets as separating $\mathbf{Q}$ (as well as the real line!) into two pieces, one to the left of the other. The "number" separating them is the corresponding real (rational or irrational) number.

Example. The rational number $1/2$ corresponds to the cut

$$A = \{r \in \mathbf{Q} \,|\, r \leq 1/2\} \text{ and } B = \{r \in \mathbf{Q} \,|\, r > 1/2\}$$

(note that all symbols used are legitimate in $\mathbf{Q}$). The irrational number $\sqrt{2}$ is a bit harder to write as a pair (A, B):

$$A = \{r \in \mathbf{Q} \,|\, r \leq 0\} \cup \{r \in \mathbf{Q} \,|\, r > 0 \text{ and } r^2 \leq 2\},$$

and

$$B = \{r \in \mathbf{Q} \,|\, r > 0 \text{ and } r^2 > 2\}$$

(again, all symbols are consistent with $\mathbf{Q}$).

Problems. 1. Define (as elegantly as possible) the Dedekind cut (A, B) that corresponds to the irrational number $1 + \sqrt[3]{2}$.

2. Defining

$$(A, B) + (C, D) = (\,\{a + c \,|\, a \in A,\ c \in C\}\,, \{b + d \,|\, b \in B,\ d \in D\})$$

(for all cuts) and

$$(A, B)\,(C, D) = (E, F)$$

where

$$E = \{r \in \mathbf{Q} \,|\, r \leq 0 \text{ or } r = ac \text{ for some } a \in A,\ c \in C\}$$

and

$$F = \{r \in \mathbf{Q} \,|\, r \text{ is not in } E\}$$

(for cuts corresponding to positive reals), show that the set of all cuts forms a field *containing the rationals*. Can you define the natural order on this field? You may assume that if (A, B) and (C, D) are cuts, then so are their sum and product... or if you are feeling adventurous, prove this as well!