Cat and Mouse Worksheet

Problem: Think of a square room with vertices $(0,0)$, $(A,0)$, (A,A), and $(0,A)$, where $A > 0$. A mouse runs down from (A,A) (starting at $t=0$) to $(A,0)$ at a constant speed of v_m units per second. A cat starts running at the origin at the same time as the mouse, also at a constant speed $v_c = Kv_m$, but following a curved path such that the cat always faces the mouse. What is the positive constant K, if the cat catches the mouse exactly at the corner $(A,0)$?

1. Draw a picture of the room and the cat's path.

2. What are the unknown but constant quantities (parameters) in this problem, other than K? For all we know, the value of the constant K may depend on these unspecified numbers!

3. Compute the time t_m it takes the mouse (and the cat) to complete its run.

$t_m =$

4. Write the position function, $\vec{\mathbf{r}}_m(t)$, of the mouse between $t=0$ and $t=t_m$ as a vector function in terms of the unit vectors $\vec{\mathbf{i}}$ and $\vec{\mathbf{j}}$.

$\vec{\mathbf{r}}_m(t) =$

5. Write the velocity function, $\vec{\mathbf{v}}_m(t)$, of the mouse between $t = 0$ and $t = t_m$ as a vector function in terms of the unit vectors $\vec{\mathbf{i}}$ and $\vec{\mathbf{j}}$.

$$\vec{\mathbf{v}}_m(t) =$$

6. The following is the most important equation in the project: we do not know the explicit position function, $\vec{\mathbf{r}}_c(t)$, of the cat. What we do know is the cat (hence its velocity vector, $\vec{\mathbf{v}}_c(t)$) always faces the mouse. We write

$$\vec{\mathbf{r}}_m(t) - \vec{\mathbf{r}}_c(t) = \gamma(t)\vec{\mathbf{v}}_c(t), \qquad (*)$$

or "the velocity vector of the cat is always a positive multiple of the difference of the position vectors of the animals". Note that we write the "multiple" $\gamma(t)$ (a name we made up) on the other side of the equation for convenience and that $\gamma(t)$ changes with time, so that it is a scalar function. Draw the vectors given in this equation for some arbitrary time t to see why this interpretation is correct.

7. Substitute $t = 0$ into Equation $(*)$ and solve for $\gamma(0)$. This is the constant of proportionality at the beginning of the chase. (You know both the direction and the length of the cat's velocity at the time $t = 0$.)

$$\gamma(0) =$$

8. While we are at it, what is $\gamma(t_m)$, or the constant of proportionality at the final and fatal moment of the chase? (Think of both position vectors.)

$$\gamma(t_m) =$$

9. All our scalar and vector functions will be assumed to be continuous on the time interval $[0, t_m]$ and differentiable on $(0, t_m)$. Differentiate both sides of Equation $(*)$ on the interval $(0, t_m)$ as a vector function, using the rules for differentiation of vector/scalar function that you thought you would never, ever need. Use the notation $\vec{\mathbf{v}}_c(t)$ for the cat's velocity and $\vec{\mathbf{a}}_c(t)$ for its acceleration. The resulting equation will be this (do the rough work, if you like, in the space provided under the equation):

$$\vec{\mathbf{v}}_m(t) - \vec{\mathbf{v}}_c(t) = \qquad (**)$$

10. At some point in the course, you must have seen that if a vector function has *constant length* (but not necessarily constant direction), then its derivative -as a vector function- is always perpendicular to the original function. Now take the dot product of both sides of $(**)$ with the cat's velocity function $\vec{\mathbf{v}}_c(t)$ (what is the length of this vector function for any t?) and finally solve for the following term (do the rough work in the space provided under the equation). The unknown scalar function $\gamma(t)$ will of course be there in some form.

$$\vec{\mathbf{v}}_m(t) \cdot \vec{\mathbf{v}}_c(t) = \qquad (***)$$

11. Substitute the notation

$$\vec{\mathbf{r}}_c(t) = x(t)\,\vec{\mathbf{i}} + y(t)\,\vec{\mathbf{j}} \quad \text{and} \quad \vec{\mathbf{v}}_c(t) = x'(t)\,\vec{\mathbf{i}} + y'(t)\,\vec{\mathbf{j}}$$

to rewrite the expression $\vec{\mathbf{v}}_m(t) \cdot \vec{\mathbf{v}}_c(t)$. Use the space provided under the new equation for rough work.

$$\vec{\mathbf{v}}_m(t) \cdot \vec{\mathbf{v}}_c(t) =$$

12. Substitute the expression you have found in Question 11 into Equation $(***)$ and solve for $y'(t)$. Use the space provided under the new equation for rough work. This is the cat's vertical velocity function.

$$y'(t) =$$

13. Integrate both sides of the Equation above to find $y(t)$, at least in the interval $(0, t_m)$. This is the cat's vertical position function. Use an arbitrary constant, C.

$$y(t) =$$

14. With our previous assumptions, the formula for $y(t)$ also holds by continuity at the ends of the time interval, so that $y(t)$ has the same formula on $[0, t_m]$. Find the actual value of the constant C by substituting $t = 0$ and solving for C. Use the value of $\gamma(0)$ computed in Question 7. (Rough work in space underneath!)

$$C =$$

15. Now rewrite $y(t)$ with the actual value of C. Use Kv_m instead of v_c.

$$y(t) = \qquad\qquad\qquad\qquad (t \in [0, t_m]).$$

Almost there!

16. Finally, the moment of truth! Substitute $t = t_m$ (the last moment of the chase) to find out what K is. Use the value of $\gamma(t_m)$ in Question 8.

$K =$

The really short solution.

Change the frame of reference: think of the animals as initially sitting on a plank of length $A\sqrt{2}$ (the length of the diagonal of the room), facing each other, where the mouse will not move at all. (The plank, let's say, is gliding on an icy surface so that the mouse end of the plank looks as if it is moving on a straight line at a constant speed of v_m with respect to a stationary observer). The cat starts moving at the constant speed of Kv_m on the plank (hence facing the mouse at all times) and covers the distance in $t_m = A/v_m$ seconds. Now use the uniform motion formula "distance equals speed times time" once more for the motion of the cat, and solve for K:

$$A\sqrt{2} = Kv_m \frac{A}{v_m}, \text{ or } K = \sqrt{2}.$$

How about that!