# Vectors from Quaternions

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**Problem:** William Rowan Hamilton (1805-1865) was able to define the four field operations on pairs $(a, b)$ of real numbers, which led to the first abstract construction of complex numbers without *a priori* assuming that they exist. (Each pair $(a, b)$ corresponds to a complex number $a + b\,\mathbf{i}$; how did he define the four operations on pairs?) Emboldened by his success, Hamilton next tried to define similar operations on triples of complex numbers that would satisfy the same properties enjoyed by real and complex numbers (such as the associativity and commutativity of addition and multiplication). This was not just an abstract exercise; he wanted the 3-D vector space (where the contemporary physics took place) to be a field. He was not successful in this attempt.

However, Hamilton did manage to define addition and multiplication on *quadruplets* $(a, b, c, d)$ of real numbers, at the expense of commutativity of multiplication. He called this new object **quaternions** and was so excited that he carved the rules for multiplication on a bridge! This 4-D real vector space was the first example of a noncommutative *division ring* (an object satisfying all properties of a field except possibly for commutativity of multiplication). Interpreting the fourth dimension as time, Hamilton introduced the basis $\{\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$. The basis element $\mathbf{1}$ represented the unit of time and the remaining basis vectors corresponded to the familiar 3-D unit vectors along the $x$-, $y$-, and $z$-axes. Multiplication among basis elements -to be extended by the distributive law- was defined by the following table:

| $\times$ | $\mathbf{1}$ | $\mathbf{i}$ | $\mathbf{j}$ | $\mathbf{k}$ |
|---|---|---|---|---|
| $\mathbf{1}$ | $\mathbf{1}$ | $\mathbf{i}$ | $\mathbf{j}$ | $\mathbf{k}$ |
| $\mathbf{i}$ | $\mathbf{i}$ | $-\mathbf{1}$ | $\mathbf{k}$ | $-\mathbf{j}$ |
| $\mathbf{j}$ | $\mathbf{j}$ | $-\mathbf{k}$ | $-\mathbf{1}$ | $\mathbf{i}$ |
| $\mathbf{k}$ | $\mathbf{k}$ | $\mathbf{j}$ | $-\mathbf{i}$ | $-\mathbf{1}$ |

Given any quaternion $a\,\mathbf{1} + b\,\mathbf{i} + c\,\mathbf{j} + d\,\mathbf{k}$, Hamilton defined $a\,\mathbf{1}$ to be the **scalar** part and $b\,\mathbf{i} + c\,\mathbf{j} + d\,\mathbf{k}$ to be the **vector** part.

**1.** Compute the product of the two quaternions below, using all familiar properties of fields (except commutativity of multiplication) and the multiplication table. The symbols $a, b, c, d, e, f, g, h$ represent real numbers.

$(a\,\mathbf{1} + b\,\mathbf{i} + c\,\mathbf{j} + d\,\mathbf{k})\,(e\,\mathbf{1} + f\,\mathbf{i} + g\,\mathbf{j} + h\,\mathbf{k}) =$

**2.** Let $a = e = 0$, so that the quaternions above are in fact "vectors". Write down the product of these two vectors. Do you recognize any familiar vector operations in the answer? If $\mathbf{u}$ and $\mathbf{v}$ are two vectors in this sense, what is the general formula for the quaternion product of two vectors, written without the "coordinates" $b, c, \ldots$?

$(b\,\mathbf{i} + c\,\mathbf{j} + d\,\mathbf{k})\,(f\,\mathbf{i} + g\,\mathbf{j} + h\,\mathbf{k}) =$

$\mathbf{u}\,\mathbf{v} =$

**3.** The only product you have seen in an undergraduate course between two 3-D vectors (that produces another vector) is probably the cross product. Recall that the cross product of two parallel vectors is the zero vector, so that

$$\mathbf{i} \times \mathbf{i} = \mathbf{0}, \quad \mathbf{j} \times \mathbf{j} = \mathbf{0}, \quad \mathbf{j} \times \mathbf{j} = \mathbf{0}.$$

Can you reconcile this fact with the rules

$$\mathbf{i}\,\mathbf{i} = -\mathbf{1}, \quad \mathbf{j}\,\mathbf{j} = -\mathbf{1}, \quad \mathbf{k}\,\mathbf{k} = -\mathbf{1}$$

for quaternions?

**4.** While you are at it, prove that the quaternions form a division algebra. You may assume associativity and distributivity. (Hint: Imitate complex numbers when looking for a multiplicative inverse.) Also, recall that the cross product is not associative; why does the associativity of quaternion multiplication not contradict this?

**Algebraic note:** Quaternions include **R** (with basis $\{\mathbf{1}\}$) and **C** (with basis $\{\mathbf{1}, \mathbf{i}\}$) as commutative subrings. Neither is an ideal. In fact, division algebras, hence fields, cannot contain any "nontrivial" left or right ideals... prove it!

**Historical note:** Hamilton's introduction of quaternions into physics was championed by James Clerk Maxwell (1831-1879), who wrote the combined field equations for the electromagnetic field (they fit on a T-shirt). In fact, Maxwell used quaternions to introduce the coordinate-free notation into physics. Despite his efforts, the new objects slowly disappeared from the physics scene. However, the vector notation and vector operations stayed with us forever, thanks to Josiah Willard Gibbs (1839-1903) and Oliver Heaviside (1850-1925), who noticed that the full algebra of quaternions was not necessary to describe physical processes.