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Project 1: Computer Graphics

1 Introduction

Computer graphics are images displayed or animated on a computer screen. Applications of computer graphics are widespread and growing rapidly. For instance, computer-aided design is an integral part of many engineering processes. The entertainment industry has made the most spectacular use of computer graphics–from the special effects in *King Kong* to the Nintendo Wii.

Most interactive computer software for business and industry makes use of computer graphics including screen displays, desktop publishing, and slide production for commercial and education presentations. Consequently, most students studying a computer language spend some time learning how to use two-dimensional (2D) graphics, and maybe three dimensional (3D) graphics.

2 Transformations in 2D

We begin by examining some of the basic mathematics used to manipulate and display graphical images such as a letter of the alphabet. Such an image consists of a number of points, connected lines or curves, and information about how to fill closed regions bounded by the lines and curves. Often, curved lines are approximated by short straight-line segments, and a figure is defined mathematically by a list of points.

The capital letter N can be determined by eight points, or vertices. The coordinates of the points can be stored in a data matrix D. In addition to D, it is necessary to specify which vertices are connected by lines, but we will omit this detail for now.

The main reason graphical objects are described by a collection of straight-line segments is that the standard transformations in computer graphics map line segments onto other line segments. Once the vertices that describe an object have been transformed, their images can be connected with the appropriate straight lines to produce the compete image of the original object.

1. On graph paper, carefully draw the letter N. Construct a data matrix D containing the eight points or vertices from your drawing. Include in your report a picture of your letter N.

2. Given a matrix $A = \begin{bmatrix} 1 & .25 \\ 0 & 1 \end{bmatrix}$, describe the effect of left multiplication by A on the letter N. Include in your report a picture of the letter N after the transformation by A.

3. In the previous question, the N looks a bit too wide after the transformation by the matrix A. to compensate, multiply (on the left) the coordinates of the letter obtained in the previous question by the matrix $S = \begin{bmatrix} .75 & 0 \\ 0 & 1 \end{bmatrix}$. Include a picture of this letter N in your report and describe with a short sentence the effect of left multiplication by the matrix S.

There are three different ways we can move an object to another location. The first way is by translation. We can *translate* points in the xy-plane to a new position by adding translation amounts to the coordinates of the points. For example, we translate the point (x, y) by moving d_x units parallel to the x-axis and by d_y units parallel to the y-axis to the new point (x', y'). Then $x' = x + d_x$ and $y' = y + d_y$.

4. Translate your original letter N by 3 units to the right and 2 units up. Include a picture of the translated letter N in your report, and describe in a short sentence the effect of this translation on the original letter N.

Points can also be *scaled* (stretched) by s_x along the x-axis and by s_y along the y-axis into a new point by the multiplications $x' = s_x \cdot x$ and $y' = s_y \cdot y$.

5. Scale your original letter N by $1/2$ in x and $1/4$ in y. Now scale the original letter N by 2 in x and 4 in y. Include a printout of both scaled letters and describe in a short sentence the effect of these transformations on the original letter N.

6. In the previous question, what matrices A and B would produce the effects of the transformations on the letter N when the matrix D is multiplied on the left by A or B?

Points can also be *rotated* through an angle θ about the origin. Let (x, y) denote a point in the plane that we wish to rotate through the angle θ, and let (x_r, y_r) denote the new point after the rotation. Our first objective is to calculate the new coordinates in terms of the old ones. Let b denote the length of the line segment from $(0, 0)$ to (x, y) and ψ denote the angle between the x-axis and this line. Then $x = b\cos(\psi)$ and $y = b\sin(\psi)$. Similar formulas give us x_r and y_r in terms of the angle $\theta + \psi$.

7. Find formulas for x_r and y_r similar to those given for x and y in terms of the angle $\theta+\psi$. Use trigonometric identities to rewrite these formulas in terms of the angles θ and ψ. Then identify occurrences of x and y in these expressions, and rewrite the formulas to give x_r and y_r in terms of x, y, and θ only.

8. Rewrite this transformation (the rotation through the angle θ) as left multiplication by a matrix A, that is, find A so that $A\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x_r \\ y_r \end{bmatrix}$. We will call A the *rotation matrix* for the angle θ.

9. Test your result from the previous question by rotation the point $(1, 0)$ through an angle of $60°$. First, use some scratch paper and trigonometry to see what the answer should be. Then let B be the 2×2 rotation matrix for the angle θ of $60°$. Apply B to the appropriate vector for the point $(1, 0)$. Check that your matrix B gives the same result as the hand calculation.

10. Repeat the preceding step with the point $(0, 1)$.

11. Using your calculator, compute B^6, B^{12}, B^{18}, and B^{24}. Explain your results in terms of rotations. In particular, how could you have predicted the particular matrices that your calculator produced?

12. Let K be the 2×2 rotation matrix for a rotation of $15°$. Compute B^2, K^4, K^2B, KBK and BK^2. Compare the results and explain what you see.

13. Explain why the product of any two 2×2 rotation matrices is another rotation matrix. Illustrate your argument with a specific example.

14. In general, we know that matrix multiplication is not commutative; i.e., if A and B are both $n \times n$ matrices, then usually AB is not the same as BA. However, rotation matrices are a special case. Explain why, for any two 2D rotation matrices Q and R, it must follow that $QR = RQ$. Then use B and K from the preceding step to illustrate your argument.

3 Homogeneous Coordinates

In Part 1, we saw that there were three types of transformations in 2D – translation, scaling, and rotation. OF these three, scaling and rotation can be accomplished by left multiplication of a 2×2 matrix. It turns out that translation **cannot** be accomplished by left multiplication of a 2×2 matrix. WE would like to be able to treat all three transformations in a consistent way, so that they can be combined easily. This can be done by expressing the points in *homogeneous coordinates*. Homogeneous coordinates were first developed in geometry and have been applied in graphics. Numerous graphics subroutine packages and display processors work with homogeneous coordinates and transformations.

Each point (x, y) in the xy-plane can be identified with the $(x, y, 1)$ on the plane in $\mathbb{R}^3$ that lies one unit above the xy-plane. We say that (x, y) has *homogeneous coordinates* $(x, y, 1)$. For example, the point $(0, 0)$ has homogeneous coordinates $(0, 0, 1)$. Homogeneous coordinates for points can be transformed via multiplication by 3×3 matrices.

1. What 3×3 matrix has the effect of translating the point (x, y) by d_x units parallel to the x-axis and by d_y units parallel to the y-axis to the new point (x', y')? That is, find a 3×3 matrix A so that $A \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix}$.

2. Repeat the previous question for scaling by s_x along the x-axis and by s_y along the y-axis.

3. Repeat for rotations through the angle θ.

The movement of a figure on a computer screen often requires two or more basic transformations. The compositions of such transformations corresponds to matric multiplication when homogeneous coordinates are used. For the next three questions below, find 3×3 matrices that produce the described composite 2D transformation, using homogeneous coordinates. Illustrate each composite 2D transformation on the image of a triangle with data matrix $D = \begin{bmatrix} 5 & 2 & 4 \\ 0 & 2 & 3 \end{bmatrix}$.

4. Translate by $(3, 1)$ and then rotation $45°$ about the origin.

5. Translate by $(-2, 3)$ and then scale the x-coordinate by .8 and the y-coordinate by 1.2.

6. Reflect points in the x-axis, and then rotate $30°$ about the origin.

7. Consider the following geometric 2D transformations: D a dilation (in which x-coordinates and y-coordinates are scaled by the same factor), R a rotation, and T a translation. Does D commute with R? In other words, is $DR = RD$ for any rotation and dilation D? Does D commute with T? Does R commute with T?

8. A rotation on a computer screen is sometimes implemented as the product of two shear-and-scale transformations, which can speed up calculations that determine how a graphic image actually appears in terms of screen pixels. (The screen consists of rows and columns of small dots, called pixels.) The first transformation A_1 translates vertically and then compresses each column of pixels; the second A_2 translates horizontally and then stretches each row of pixels, where $A_1 = \begin{bmatrix} 1 & 0 & 0 \\ \sin(\phi) & \cos(\phi) & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $A_2 = \begin{bmatrix} \sec(\phi) & -\tan(\phi) & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Using $\phi = 30°$, apply A_1 followed by A_2 to the rectangle with data matrix $D = \begin{bmatrix} 1 & 1 & 4 & 4 \\ 1 & 3 & 1 & 3 \end{bmatrix}$.

Show, for a given angle ϕ, that the composition of the two transformations is a rotation in 2D.

4 3D Computer Graphics

Some of the newest and most exciting work in computer graphics is connected with molecular modeling. With 3D graphics, a biologist can examine a simulated protein molecule and search for active sites that might accept a drug molecule. The biologist can rotate and translate an experimental drug and attempt to attach it to the protein. This ability to visualize potential chemical reactions is vital to modern drug and cancer research. In fact, advances in drug design depend, on some extent, upon progress in the ability of computer graphics to construct realistic simulations of molecules and their interactions.

Current research in molecular modeling is focused on *virtual reality*, an environment in which a researcher can see and feel the drug molecule slide into the protein. Another design for virtual reality involves a helmet and glove that detects head, hand, and finger movements. The helmet contains two tiny computer screens, one for each eye. Making this virtual environment more realistic is a challenge to engineers, scientist, and mathematicians. The mathematics we will examine barely opens the door to this interesting and important field of research.

In 3D, a point can be rotated about any one of the three axes. The 3d version of the 2d rotation matrix A are given by: $P_\theta = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $Q_\theta = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{bmatrix}$, and $R_\theta = \begin{bmatrix} \cos(\theta) & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) \end{bmatrix}$.

The matrix P_θ rotates a point in 3D about the z-axis, Q_θ about the x-axis, and R_θ about the y-axis. These are not the only possible rotations in 3D, but we will limit our attention to these possibilities for now.

1. What feature of each of the matrices above tells us quickly the axis about which the rotation is being done?

2. We saw earlier that multiplication of 2D rotation matrices is commutative, even though matrix multiplication in general is not commutative. We'll try this for 3D rotation matrices. Let P_{30} and P_{45} be the matrices for rotations of 30° and 45°, respectively, around the z-axis. Compute $P_{30}P_{45}$ and $P_{45}P_{30}$. What do you observe?

3. Now compute $P_{30}R_{45}$ and $R_{45}P_{30}$. what do you observe? Hold an object (such as your textbook) out in front of you, and rotate it as indicated in these product matrices. Try to convince yourself that what you observed mathematically is consistent with reality. Write a couple of sentences to describe what you observed, both physically and mathematically.

4. Try to generalize what you computed in the preceding steps. In particular, are any 3D rotation matrices multiplicatively commutative? If so, which ones?

5. Suppose an image is stored in computer memory as we described earlier–a set of coordinates in 3D space. Assume that when the object is displayed on the screen, the x-axis is perpendicular to the screen, the y-axis is horizontal, and the z-axis is vertical. Thus, the yz-plane is on the surface of the screen. The software can perform rotations by multiplying each point (as a vector) by an appropriate rotation matrix and then displaying the result. If we want to display the object so that it is first flipped over from our right to our left, and then the axis projected toward us is tilted upward $20°$, what matrix can the computer use to do this transformation? Explain your reasoning, and compute the matrix.

5 Homogeneous Coordinates in 3D

Just as 2D transformation can be represented by 3×3 matrices, using homogeneous coordinates, 3D transformations can be represented by 4×4 matrices, providing we use homogeneous coordinate representations of points in 3-space as well. By analogy with the 2D case, we say that $(x, y, z, 1)$ are homogeneous coordinates for the point (x, y, z).

In the first four questions below, give the 4×4 matrix for performing the indicated translations in 3D.

1. Rotation about the y-axis through an angle of $30°$.

2. Translation by $(-6, 4, 5)$.

3. Scaling by $(-6, 4, 5)$.

4. Rotation about the z-axis through an angle of $-30°$, and then translates by $(5, -2, 1)$.

5. Illustrate the effects of the above transformations on the triangle with vertices $(4.2, 1.2, 4)$, $(6, 4, 2)$, and $(2, 2, 6)$.