Cardinality of a set

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- By definition: two sets have the same *cardinality* if and only if there is a bijective function from one to the other.
- Any set that is in 1-1 correspondence with $\{1, 2, \ldots, n\}$ for some positive integer *n* is said to have (finite) cardinality n. Card(ϕ) is zero.
- The cardinality of a set is said to be *infinite* if and only if it is not finite. OR: The cardinality of a set is infinite if and only if a bijection between this set and a proper subset of it can be found. (Example: **Z** and 2**Z**)
- Let us show the cardinality of sets $A, B, \ldots$ with Greek letters $\alpha, \beta, \ldots$ respectively. As we build new sets out of old (e.g. by taking unions or Cartesian products) we will want to know what the cardinality of the new set is. But first, let us look at the first two infinite cardinal numbers.
- Any set with a bijection to **N**= $\{1, 2, \ldots, n, \ldots\}$ is said to have (infinite) cardinality χ_0 (aleph-naught). Such sets are also called *countable* or *denumerable*. Any larger cardinality will be *uncountable* (to varying degrees!). Examples of countable sets are **Z**, **Q**, **Q**×**Q**.
- The "next" cardinality, χ_1 (aleph-one), is that of real numbers. The guess that there is no other cardinality between χ_0 and χ_1 is called the "continuum hypothesis". It is impossible to write the real numbers (even those in a small interval) as a sequence.
- To get larger and larger cardinalities, one takes the power set of the previous set (i.e. the set of all subsets). For example, it can be shown that the *power set* of **N** has cardinality χ_1. In general, Card(A)<Card(P(A)).
- If Card$(A) = \alpha$ and Card$(B) = \beta$, with A, B disjoint sets, then Card$(A \cup B) = \alpha + \beta$. Similarly, Card$(A \times B) = \alpha\beta$.
- Then we need to know how to add and multiply cardinalities. There exists an order relationship among such entities. For example, we have $0 < 1 < 2 < \cdots < \chi_0 < \chi_1 < \cdots$.
- If α is infinite and $\beta \leq \alpha$, then $\alpha + \beta = \alpha$ and $\alpha\beta = \alpha$. In addition or multiplication, the larger cardinality dominates. (Do you remember writing $\infty + 5 = \infty$ in Calculus I? Same idea.) Examples: $\chi_0 + \chi_1 = \chi_0\chi_1 = \chi_1 + \chi_1 = \chi_1\chi_1 = \chi_1$.
- Another Theorem: if A is any set with cardinality α, then the union $A \cup (A \times A) \cup (A \times A \times A) \cup \cdots$ has cardinality $\chi_0\alpha$. (If this were a finite union, it would have been easy to derive this result from the basics above!)

1. Prove that **Z** is countable.
2. Prove that **Q** is countable. (Hint: Try to arrange all positive rationals in an infinite table, with possible redundancies, starting from a corner.)
3. Prove that **R** is not countable. (Hint: Suppose the real numbers between 0 and 1 *are* countable… then put them in a sequence, writing appropriate symbols for the digits, and show that it is possible to construct a real number in the same interval which is *not* in this sequence.)
4. Show that there are as many polynomials with integer coefficients as there are integers. Use the properties above.