How many factors does an integer have?

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Problem: 1. Let $n \geq 2$ be an integer. Find a formula for the number of positive factors (divisors) of n, including 1 and itself. (Hint: Use explicit unique factorization into positive primes.)

2. Using the above result, prove that such an integer is a square if and only if it has an odd number of positive factors. (Question from my then 8-year-old son.)

3. Now prove the statement in Problem 2 without using the formula for the number of positive factors. (Hint: List the factors in both ascending and descending orders.) The answer was supplied by my students, who turned out to be smarter and more practical-minded than their instructor.

4. If a positive integer n has 42 factors, what are the possibilities for the number of distinct primes in its prime factorization? What prime powers are possible for each case?

5. Prove that there exists a positive integer n with k distinct prime factors, with powers $e_1, \ldots, e_k$, for every sequence of integers $k, e_1, \ldots, e_k \geq 1$ (pretty easy, but you need to use a well-known theorem to justify your answer). Argue that there are in fact infinitely many n with this property.