Topic: Improper Integrals

A definite integral is considered improper if either the function goes undefined in the domain of integration or one (or both) of the endpoints of integration is infinite. Both types of improper integrals are handled using limits. The integral is said to converge/diverge if the corresponding limit converges/diverges

Examples:

1. Determine whether the integral $\int_0^3 \frac{1}{\sqrt{3-x}}\,dx$ converges or diverges. If it converges, find its value.

The function $f(x) = \frac{1}{\sqrt{3-x}}$ is undefined at $x = 3$ because of division by 0. Therefore, this is an improper integral. We handle it with limits as follows:

$$\int_0^3 \frac{1}{\sqrt{3-x}}\,dx = \lim_{t\to 3^-} \int_0^t \frac{1}{\sqrt{3-x}}\,dx$$

The antiderivative of $f(x)$ is $-2\sqrt{3-x}$, so we get

$$\lim_{t\to 3^-} \int_0^t \frac{1}{\sqrt{3-x}}\,dx = \lim_{t\to 3^-} -2\sqrt{3-x}\Big|_0^t = \lim_{t\to 3^-} -2\sqrt{3-t} + 2\sqrt{3-0} = 2\sqrt{3}$$

Thus, the integral converges to $2\sqrt{3}$.

2. Determine whether the integral $\int_{-2}^3 \frac{1}{x^3}\,dx$ converges or diverges. If it converges, find its value.

The function $f(x) = \frac{1}{x^3}$ goes undefined at $x = 0$. Since the point of discontinuity lies in the middle of the region of integration, we must split the integral into two pieces and use two limits. We break the integral up as follows:

$$\int_{-2}^3 \frac{1}{x^3}\,dx = \int_{-2}^0 \frac{1}{x^3}\,dx + \int_0^3 \frac{1}{x^3}\,dx = \lim_{t\to 0^-} \int_{-2}^t \frac{1}{x^3}\,dx + \lim_{t\to 0^+} \int_t^3 \frac{1}{x^3}\,dx$$

Note that if either of the limits on the right diverges, then the original integral is said to diverge. The antiderivative of $f(x)$ is $-\frac{1}{2x^2}$, so for the first limit we get

$$\lim_{t\to 0^-} \int_{-2}^t \frac{1}{x^3}\,dx = \lim_{t\to 0^-} -\frac{1}{2t^2}\Big|_{-2}^0 = \lim_{t\to 0^-} -\frac{1}{2t^2} + \frac{1}{8} = -\infty$$

Since the first integral was divergent, the original integral is also divergent.

3. Determine whether the integral $\int_1^\infty \frac{1}{x^2}\,dx$ converges or diverges. If it converges, find its value.

This integral is improper because one of the endpoints of integration is infinite. We handle it with limits as follows:

$$\int_1^\infty \frac{1}{x^2}\,dx = \lim_{t\to\infty}\int_1^t \frac{1}{x^2}\,dx$$

The antiderivative of $f(x)$ is $-\dfrac{1}{x}$, so we get

$$\lim_{t\to\infty}\int_1^t \frac{1}{x^2}\,dx = \lim_{t\to\infty} -\frac{1}{t^2}\Big|_1^\infty = \lim_{t\to\infty} -\frac{1}{t^2} + 1 = 1$$

So the original integral converges to 1.