

**Mathematical Abundance:
Designs, Graphs and Number
Honoring the 72nd birthday of
Charles Vanden Eynden
Illinois State University, April 18–19, 2008**



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A quote from Jean-Pierre Bourguignon:

... geometry evolved from uniqueness to abundance.



Matchings in Bipartite Graphs:

Properties and Structure



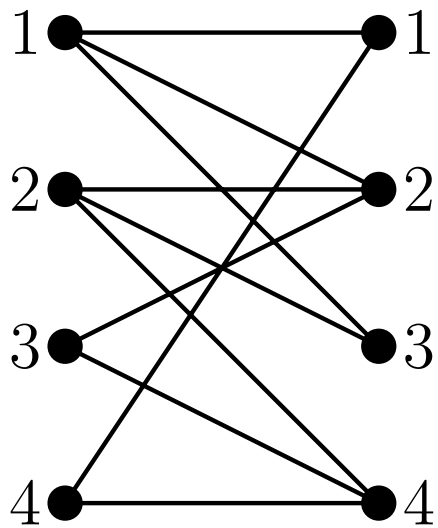
Matchings in Bipartite Graphs: Properties and Structure

Richard A. Brualdi

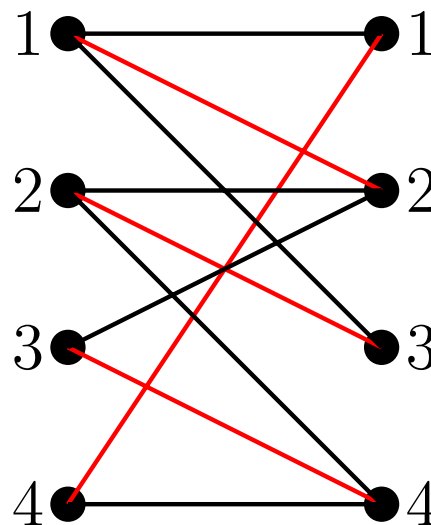
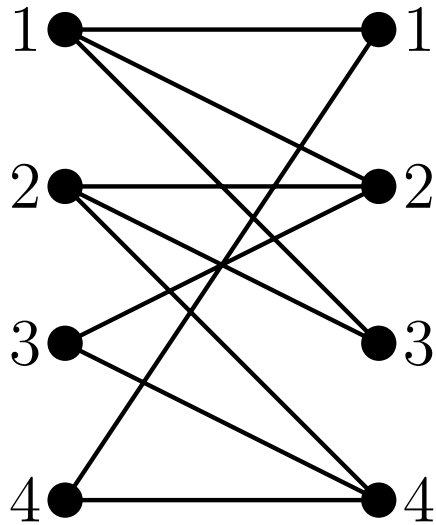
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perfect matching

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Label the edges (i.e. label the 1s in the matrix) as a way to distinguish perfect matchings:

$$\begin{bmatrix} a & \mathbf{b} & c & 0 \\ 0 & d & \mathbf{e} & f \\ 0 & g & 0 & \mathbf{h} \\ \mathbf{i} & 0 & 0 & j \end{bmatrix} \cdot$$

The collection of perfect matchings for this bipartite graph/matrix

$$\begin{bmatrix} a & b & c & 0 \\ 0 & d & e & f \\ 0 & g & 0 & h \\ i & 0 & 0 & j \end{bmatrix}.$$

is:

$$\{a, e, g, j\}, \{b, e, h, i\}, \{c, d, h, i\}, \{c, f, g, i\}.$$



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This is the **perfect matching hypergraph** $\mathcal{M}(G)$ of G with vertices a, b, \dots, j and hyperedges given by the perfect matchings. If $G \subseteq K_{n,n}$, then all hyperedges have cardinality n .

Let $G \subseteq K_{n,n}$ be a bipartite graph with labeled edges, that is, let A be a matrix of order n whose nonzero entries are labeled with different symbols $V = \{a, b, c, \dots\}$. The **perfect matching hypergraph** $\mathcal{M}(A)$ of A has vertex set V and edges (of size n) corresponding to the **perfect matchings** of G , or **diagonals** of A .

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General Assumption: Every edge of G is in a perfect matching (every nonzero entry of A belongs to a diagonal) and G is connected; the matrix A is **fully indecomposable**.

Let B be another matrix of order n with the same number of nonzero entries as A .

Then A and B have **isomorphic diagonal hypergraphs** provided the nonzero entries of B can also be labeled with the elements of V so that their perfect matching hypergraphs are identical:

$$\mathcal{M}(A) = \mathcal{M}(B).$$

The resulting bijection between their nonzero entries is a **perfect matching preserver**.



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Note that in terms of the biadjacency matrix A , if $B = PAQ$ or $B = PA^TQ$, then we have an isomorphism of the bipartite graphs.

But G and H can have isomorphic matching hypergraphs without being isomorphic graphs.



Example:

$$A = \begin{bmatrix} 0 & a & b & 0 & 0 \\ c & d & e & 0 & 0 \\ f & 0 & g & h & i \\ 0 & 0 & j & 0 & k \\ 0 & 0 & 0 & l & m \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & a & b & 0 & 0 \\ c & d & e & 0 & 0 \\ f & 0 & g & j & 0 \\ 0 & 0 & h & 0 & l \\ 0 & 0 & i & k & m \end{bmatrix}$$

have the same perfect matchings:

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so we have a **perfect matching preserver**. These bipartite graphs (or their line graphs) are not isomorphic, since one has a vertex of degree 5 and the other does not.



Example revisited:

$$A = \begin{bmatrix} 0 & a & \mathbf{b} & 0 & 0 \\ c & d & \mathbf{e} & 0 & 0 \\ f & 0 & \mathbf{g} & \mathbf{h} & \mathbf{i} \\ 0 & 0 & j & 0 & k \\ 0 & 0 & 0 & l & m \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & a & \mathbf{b} & 0 & 0 \\ c & d & \mathbf{e} & 0 & 0 \\ f & 0 & \mathbf{g} & j & 0 \\ 0 & 0 & \mathbf{h} & 0 & l \\ 0 & 0 & \mathbf{i} & k & m \end{bmatrix}$$

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have the same nonzero diagonals:

$$\{a, c, g, k, l\}, \{a, c, j, h, m\}, \{a, c, j, i, l\}, \{a, f, e, k, l\},$$

so we have a perfect matching preserver not resulting from a row or column permutation or transposition, or composition of such.



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$$\left[\begin{array}{c|c|c} A_1 & \alpha & O \\ \hline \gamma & ** & \beta \\ \hline O & \delta & A_2 \end{array} \right] \rightarrow \left[\begin{array}{c|c|c} A_1 & \alpha & O \\ \hline \gamma & ** & \delta^T \\ \hline O & \beta^T & A_2^T \end{array} \right]$$

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Note:

- (1) Transposition results when the matrix A_1 is empty.
- (2) Like row and column permutations, a partial transposition preserves perfect matchings.
- (3) A partial transposition requires a p by q zero submatrix and a complementary q by p zero submatrix where $p + q = n - 1$.

RAB and J. Ross Conjecture 1981: Every perfect matching preserver between matrices A and B of order n can be accomplished by a sequence of partial transpositions, followed by row and column permutations.



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This was proved under the assumption that the labels of n nonzero positions of A are the labels of n nonzero positions in a row or column of B . Recall the example:

$$A = \begin{bmatrix} 0 & a & b & 0 & 0 \\ c & d & e & 0 & 0 \\ f & 0 & g & h & i \\ 0 & 0 & j & 0 & k \\ 0 & 0 & 0 & l & m \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & a & b & 0 & 0 \\ c & d & e & 0 & 0 \\ f & 0 & g & j & 0 \\ 0 & 0 & h & 0 & l \\ 0 & 0 & i & k & m \end{bmatrix}$$

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$$A = \begin{bmatrix} a & b & c & 0 & 0 & 0 \\ d & e & f & 0 & 0 & 0 \\ 0 & 0 & k & l & m & 0 \\ 0 & 0 & r & s & t & 0 \\ w & 0 & 0 & 0 & u & v \\ z & 0 & 0 & 0 & x & y \end{bmatrix} \quad \text{and} \quad A^* = \begin{bmatrix} a & b & c & 0 & 0 & 0 \\ d & e & f & 0 & 0 & 0 \\ 0 & 0 & u & v & w & 0 \\ 0 & 0 & x & y & z & 0 \\ m & 0 & 0 & 0 & k & l \\ t & 0 & 0 & 0 & r & s \end{bmatrix}$$

have the same set of 16 perfect matchings.

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So the conjecture is false.



The preceding example is an example of what we call a **bi-transposition**:

$$\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \leftarrow A_1, p \text{ by } p + 1$$

$$\begin{bmatrix} k & l & m \\ r & s & t \end{bmatrix} \leftarrow A_2, q \text{ by } q + 1$$

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A_2 and A_3 are switched.



Example again:

$$\begin{bmatrix} a & b & c & 0 & 0 & 0 \\ d & e & f & 0 & 0 & 0 \\ 0 & 0 & k & l & m & 0 \\ 0 & 0 & r & s & t & 0 \\ w & 0 & 0 & 0 & u & v \\ z & 0 & 0 & 0 & x & y \end{bmatrix} \rightarrow \begin{bmatrix} a & b & c & 0 & 0 & 0 \\ d & e & f & 0 & 0 & 0 \\ 0 & 0 & u & v & w & 0 \\ 0 & 0 & x & y & z & 0 \\ m & 0 & 0 & 0 & k & l \\ t & 0 & 0 & 0 & r & s \end{bmatrix}$$

is a perfect matching preserver.

RAB, LoebI, Pangrác 2006:

Let A and B be fully indecomposable matrices of order n , that is, matching-covered, connected bigraphs $\subseteq K_{n,n}$. Suppose there is a bijection ϕ between the nonzero positions of A and those of B that gives a bijection between the perfect matchings of A and those of B . Then ϕ results from a sequence of

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There is a **digraph** $D(A)$ associated with A (or G) that plays an important role. The **vertices** are $1, 2, \dots, n$. The **(directed) edges** are those pairs (i, j) with $i \neq j$ and $a_{ij} = 1$.



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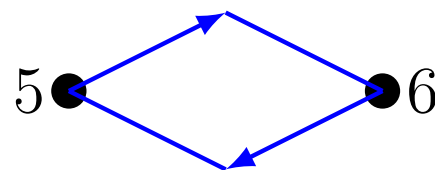
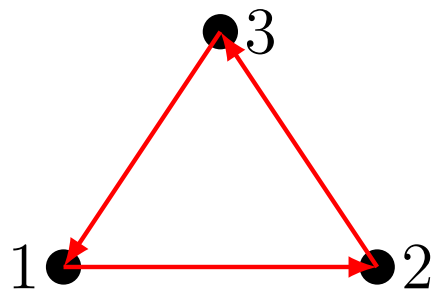
As the example illustrates there is a one-to-one correspondence between the 1-factors of $G(A)$ and unions of pairwise disjoint cycles (**cycle decomposition of a permutation**) of $D(A)$.

Another example:

$$\begin{bmatrix} 1 & \mathbf{1} & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & \mathbf{1} & 0 & 1 & 1 & 0 \\ \mathbf{1} & 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 & \mathbf{1} & 1 \\ 0 & 1 & 0 & 1 & \mathbf{1} & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & \mathbf{1} & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & \mathbf{1} & 0 & 1 & 1 & 0 \\ \mathbf{1} & 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & \mathbf{1} & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 & \mathbf{1} & 1 \\ 0 & 1 & 0 & 1 & \mathbf{1} & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 & \mathbf{1} \end{bmatrix}$$

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Recall that if D and D' are two digraphs, then a bijection $f : V(D) \rightarrow V(D')$ is an **isomorphism (resp., anti-isomorphism)** provided (u, v) is an edge in D if and only if $(f(u), f(v))$ **(resp., $(f(v), f(u))$)** is an edge of D' . **(Anti-) isomorphisms preserve cycles (as sets of edges).**

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Thomassen 1989: Let D and D' be strongly connected digraphs (where the underlying graph of D is 2-connected). Let $\varphi : E(D) \rightarrow E(D')$ be a bijection such that both φ and φ^{-1} preserve cycles. Then there is a sequence of twists that replaces D with a digraph D^* such that φ is induced by an isomorphism or anti-isomorphism of D^* onto D' .



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Let D be a digraph obtained from two disjoint digraphs D_1 and D_2 with u_1, v_1 vertices of D_1 and u_2, v_2 vertices of D_2 (where there are no edges in D_2 between u_2 and v_2) by identifying u_1 and u_2 as vertex u and v_1 and v_2 as vertex v .



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Let D^* be obtained from D_1 and D_2 by switching the directions of all arcs of D_2 and identifying u_1 and v_2 to u and identifying v_1 and u_2 to v . Thus, D^* is obtained as follows: For vertices $w \neq u_2, v_2$ in D_2 , we replace arcs $(u, w), (w, u), (v, w), (w, v)$ by arcs $(w, v), (v, w), (w, u), (u, w)$, and reverse the directions of all other arcs in D_2 but not D_1 .

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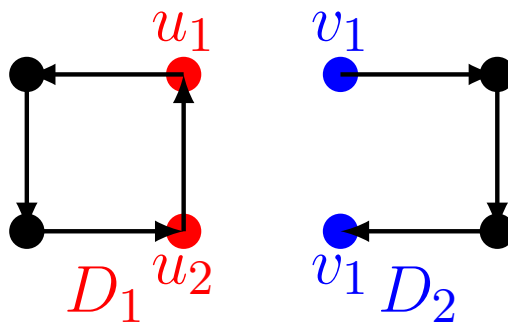
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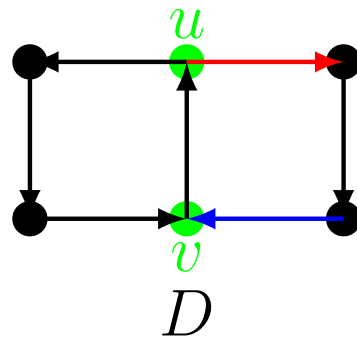
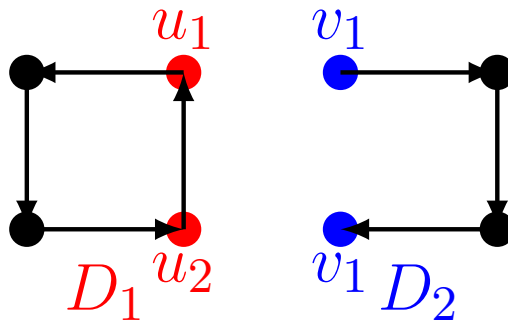
A twist preserves cycles.



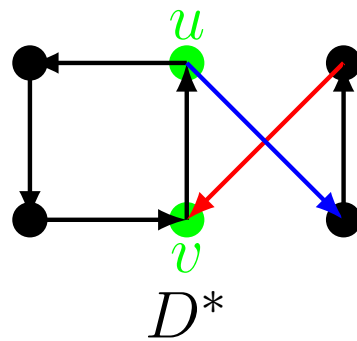
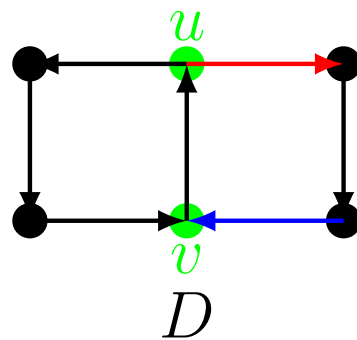
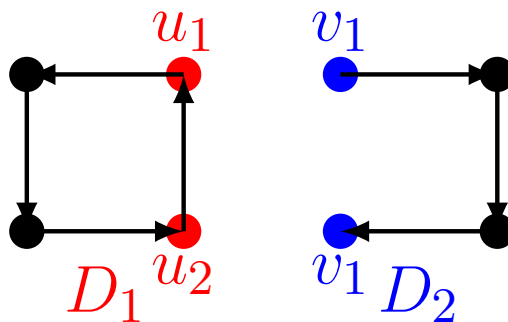
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Example



Characterization of cycle preservers (Thomassen 1989): Let D and D' be strongly connected digraphs (where the underlying graph of D is 2-connected). Let $\varphi : E(D) \rightarrow E(D')$ be a bijection such that both φ and φ^{-1} preserve cycles. Then there is a sequence of twists that replaces D with a digraph D^* such that φ is induced by an isomorphism or anti-isomorphism of D^* onto D' .

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For preservers of disjoint unions of cycles, in addition to twists, one needs a transformation we call **supertranspositions**, and this characterization of disjoint cycle preservers leads to the characterization given for perfect matching preservers.

Characterization of perfect matching preservers: Let A and B be fully indecomposable matrices of order n . Suppose there is a bijection ϕ between the nonzero positions of A and those of B that gives a bijection between the perfect matchings of A and those of B . Then ϕ results from a sequence of

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$$\begin{bmatrix} 1/4 & 1/4 & 1/2 \\ 1/4 & 1/4 & 1/2 \\ 1/2 & 1/2 & 0 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} + \frac{1}{4} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \\ + \frac{1}{4} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} + \frac{1}{4} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} .$$

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The doubly stochastic matrices of order n form a convex polytope Ω_n of dimension $(n - 1)^2$.



Matrices with **total support** (i.e. **matching covered bipartite graphs**) are geometric objects. **Faces** of Ω_n correspond to the **$(0, 1)$ -matrices $A = [a_{ij}]$ of order n with total support:**

$$A \leftrightarrow \mathcal{F}(A) = \{X = [x_{ij}], X \in \Omega_n, x_{ij} \leq a_{ij}\}.$$

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Extreme points are the permutation matrices of order n (the perfect matchings of the corresponding bigraph) with P and Q determining an edge (1-dim. face) iff, with respect to P, Q has a unique cycle of length ≥ 2 , that is $P^{-1}Q$ has a unique cycle of length ≥ 2 .

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If A is fully indecomposable, then $\dim \mathcal{F}(A) = \#(A) - 2n + 1$.



Example: $A = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix}$, **fully indecomposable,**

9 diagonals (9 perfect matchings).



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$\mathcal{F}(A)$ is an 8-dimensional simplex in \mathfrak{R}^{25} .

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Equivalently, the set $\mathcal{M}(G)$ of perfect matchings of a bipartite graph $G \subseteq K_{n,n}$ is **isolated** provided that

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Characterization 1: (J. Shao, L.-Z Ren, and Q.Wu 2006/07):
WLOG assume that $I_n \leq A$. Then $\mathcal{D}(A)$ is an isolated set of diagonals iff the following two conditions are satisfied:



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$$1 \rightarrow 2 \rightarrow 3 \rightarrow \cdots \rightarrow k \rightarrow 1$$

along with “path-chords”

$$1 \rightarrow \cdots \rightarrow p \text{ and } q \rightarrow \cdots \rightarrow r$$

where $3 \leq p < q < r \leq k$.



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- A has property N , and cannot be contracted to J_3 .
- There exists an integer p with $1 \leq p \leq n - 2$ and permutation matrices P and Q such that

$$PAQ = \begin{bmatrix} A_3 & A_1 \\ A_2 & O \end{bmatrix}$$

where A_3 is $n - p$ by $p + 1$, O is p by $n - p - 1$, and A_1 and A_2^T are vertex-edge incidence matrices of trees.

- The face $\mathcal{F}(A)$ of Ω_n is a simplex.



- **Property N : For each pair X_1 and X_2 of complementary square submatrices of A , $PAQ = \begin{bmatrix} X_1 & * \\ * & X_2 \end{bmatrix}$, where $\text{per } X_1 \leq 1$ or $\text{per } X_2 \leq 1$ (X_1 or X_2 has at most one diagonal).**

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- **Contraction:**

$$\left[\begin{array}{c|c} 1 & \alpha \\ \hline 1 & \beta \\ 0 & \\ \vdots & B \\ 0 & \end{array} \right] \rightarrow \begin{bmatrix} \alpha + \beta \\ B \end{bmatrix}.$$

Special Form:

- There exists an integer p with $1 \leq p \leq n - 2$ and permutation matrices P and Q such that

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Remark: The only other faces of Ω_n with this property are $\mathcal{F}(A)$ where A either equals J_3 (the matrix of all 1s of order 3, that is, $K_{3,3}$) or A can be contracted to J_3 .

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where A_3 is $n - p$ by $p + 1$, O is p by $n - p - 1$, and A_1 and A_2^T are vertex-edge incidence matrices of trees. **implies that there is a 1-1 correspondence between diagonals of A and the 1's in A_3 and hence $\mathcal{D}(A)$ is isolated.**

Not Property N implies that **there exist X_1 and X_2 , complementary square submatrices of A , such that $\text{per } X_1 \geq 2$ and $\text{per } X_2 \geq 2$. This gives 4 diagonals of A none of which contains an entry not in any of the other 3 diagonals. So $\mathcal{D}(A)$ is not isolated.**



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Thus we have the equivalences:



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What is the connection between the special form and this digraph characterization?



Special form implies that A has a zero submatrix $O_{p,n-p-1}$. With the assumption that $I_n \leq A$, the row indices and column indices of $O_{p,n-p-1}$ must be disjoint, and by **simultaneous permutations** (so digraph is the same), we can assume

$$A = \begin{bmatrix} A_2 & O_{p,n-p-1} \\ A_3 & A_1 \end{bmatrix} \quad (A_2 \text{ is } p \times p + 1, A_1 \text{ is } n - p \times n - p - 1).$$

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A_3 has a 1 in its upper right corner (position $(p+1, p+1)$ of A).
 Crossing out its row and column (so crossing out last column of A_2 and first row of A_1) gives A'_2 (order p) and A'_1 (order $n-p-1$)
 with $I_p \leq A'_2$ and $I_{n-p-1} \leq A'_1$.

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Thus the digraph is free-cyclic and there is a vertex belonging to all cycles of length ≥ 2 .

Example:

$$\begin{bmatrix} A_2 & O_{p,n-p-1} \\ A_3 & A_1 \end{bmatrix} = \left[\begin{array}{cccc|cccc} \mathbf{1} & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & \mathbf{1} & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & \mathbf{1} & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & \mathbf{1} & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & \mathbf{1} & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & \mathbf{1} & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & \mathbf{1} \end{array} \right]$$

$(n = 8, p = 3)$

Some references

1. R.A. Brualdi and P.M. Gibson: Convex polyhedra of doubly stochastic matrices I., *J. Combin Theory* 22 (1977), 194–230.
2. J.J. McDonald, D.D. Olesky, M.J. Tsatsomeros, and P. van den Driessche, Ray patterns of matrices and nonsingularity, *Linear Algebra Appl.* 267 (1997), 359–373.
3. J.-Y. Shao, L.-Z Ren, and Q. Wu, Characterizations and applications of the isolated set of permutations, *Linear Multilinear Alg.*, 55 (2007), 45–56.
4. R.A. Brualdi and J.-Y. Shao, Isolated sets of diagonals, diagonal hypergraphs, and simplices of doubly stochastic matrices, to appear in *Linear Multilin. Alg.*
5. R.A. Brualdi, M. Loeb, and O. Pangrác: Perfect matching preservers, *Electr. J. Combinatorics*, R95, 13(1), 2006.