Take a moment to calculate the following integral:

\[ \int e^{-x^2} \, dx \]

You should have found that this integral that appears quite straightforward is a rather challenging integral to evaluate. We do not know an antiderivative of \( y = e^{-x^2} \), there is no integral pattern like this in the table of integrals found at the back of our textbook, and the TI-89 CAS calculator returns the same expression you type in when you ask it to evaluate this integral.

This leads us to yet another integration technique, one that is applied to definite integrals: Integration Approximation.

Here we discuss five different methods for approximating the value of a definite integral. Each method revolves around associating a definite integral with area under a curve. The first three use areas of rectangles, the fourth uses areas of trapezoids, and the final approximation technique uses areas of shapes that include a portion of a parabola. The accuracy of the approximation techniques improves as we proceed.

We first need a bit of common terminology and symbolism that we will apply in all the methods we develop.

- For the methods we explore, we will subdivide into \( n \) equal parts the \( x \)-axis region between the lower and upper limits of integration. We call each of these equal parts a subdivision or subinterval.
- We label the subdivision endpoints \( x_i \), beginning with \( x_0 \), the left-most endpoint and ending with \( x_n \), the right-most endpoint. We also will refer to the left-most endpoint as \( a \) and the right-most endpoint as \( b \), corresponding to the typical use of those symbols for lower and upper limits of integration. Thus, \( x_0 = a \) and \( x_n = b \).
- We represent the length of each subdivision with \( \Delta x \) and calculate \( \Delta x \) as follows:

\[ \Delta x = \frac{b - a}{n} \]
• We will use sigma notation to represent the addition of related terms. Recall that \( \sum_{i=1}^{n} x_i \) expands to \( x_1 + x_2 + \cdots + x_{n-1} + x_n \).

Suppose we want to approximate the value of the definite integral \( \int_{0}^{1} e^{-x^2} \, dx \) and that we begin with \( n = 4 \) subdivisions. From the definite integral, we have \( x_0 = a = 0 \) and \( x_4 = b = 1 \). From this,

\[
\Delta x = \frac{b - a}{n} = \frac{1 - 0}{4} = \frac{1}{4}. \]

With each subdivision having size \( \frac{1}{4} \), we know that \( x_1 = \frac{1}{4}, x_2 = \frac{1}{2}, \) and \( x_3 = \frac{3}{4} \). Here is a graphical representation of what we have so far:

![Graphical representation](image)

This subdivision of the \( x \)-axis and the naming and labeling of points will be the same for all the approximation techniques we develop here.

**Left-Endpoint Approximation**

On each of the four subintervals shown above, we create a rectangle whose width is the length of the subdivision and whose height is determined by the function value at the left endpoint of each subdivision.

- width: \( \Delta x \), height: \( f(x_0) = f(0) \)
- width: \( \Delta x \), height: \( f(x_1) = f(\frac{1}{4}) \)
The sum of the areas of the four rectangles represents our approximation for the area under the curve and therefore represents an approximation for the value of the definite integral:

\[
\int_{0}^{1} e^{-x^2} \, dx = \Delta x \cdot f(x_0) + \Delta x \cdot f(x_1) + \Delta x \cdot f(x_2) + \Delta x \cdot f(x_3) \\
= \Delta x \left( f(x_0) + f(x_1) + f(x_2) + f(x_3) \right) \\
= \Delta x \sum_{i=0}^{3} f(x_i)
\]

This same sequence of steps can be generalized for left-endpoint approximation of the definite integral \( \int_{a}^{b} f(x) \, dx \) using \( n \) subdivisions:

\[
\int_{a}^{b} f(x) \, dx = \Delta x \cdot f(x_0) + \Delta x \cdot f(x_1) + \cdots + \Delta x \cdot f(x_{n-2}) + \Delta x \cdot f(x_{n-1}) \\
= \Delta x \left( f(x_0) + f(x_1) + \cdots + f(x_{n-2}) + f(x_{n-1}) \right) \\
= \Delta x \sum_{i=0}^{n-1} f(x_i)
\]

**Right-Endpoint Approximation**

Again we create rectangles whose widths are each the length of a subdivision, but here each height is determined by the function value at the right endpoint of each subinterval.
The sum of the areas of these four rectangles represents a right-endpoint approximation for the area under the curve and therefore is an approximation for the value of the definite integral:

\[
\int_0^1 e^{-x^2} \, dx \approx (\Delta x \cdot f(x_1)) + (\Delta x \cdot f(x_2)) + (\Delta x \cdot f(x_3)) + (\Delta x \cdot f(x_4))
\]

\[
\approx \Delta x \left( f(x_1) + f(x_2) + f(x_3) + f(x_4) \right)
\]

\[
\approx \Delta x \sum_{i=1}^{4} f(x_i)
\]

This same sequence of steps can be generalized for right-endpoint approximation of the definite integral \( \int_a^b f(x) \, dx \) using \( n \) subdivisions:

\[
\int_a^b f(x) \, dx = (\Delta x \cdot f(x_1)) + (\Delta x \cdot f(x_2)) + \cdots + (\Delta x \cdot f(x_{n-1})) + (\Delta x \cdot f(x_n))
\]

\[
\approx \Delta x \left( f(x_1) + f(x_2) + \cdots + f(x_{n-1}) + f(x_n) \right)
\]

\[
\approx \Delta x \sum_{i=1}^{n} f(x_i)
\]

Midpoint Approximation

For a third time we create rectangles each of whose width is the length of the subdivision, but now each height is determined by the function value at the midpoint of each subdivision.

\[
\text{width: } \Delta x, \text{ height: } f(\frac{x_0 + x_1}{2}) = f(\frac{1}{8})
\]
The sum of the areas of these four rectangles represents a midpoint approximation for the area under the curve and therefore is another approximation for the value of the definite integral:

\[
\int_0^1 e^{-x^2} \, dx \approx \left( \Delta x \cdot f\left( \frac{x_0 + x_1}{2} \right) \right) + \left( \Delta x \cdot f\left( \frac{x_1 + x_2}{2} \right) \right) + \\
\left( \Delta x \cdot f\left( \frac{x_2 + x_3}{2} \right) \right) + \left( \Delta x \cdot f\left( \frac{x_3 + x_4}{2} \right) \right)
\]

\[
= \Delta x \left( f\left( \frac{x_0 + x_1}{2} \right) + f\left( \frac{x_1 + x_2}{2} \right) + f\left( \frac{x_2 + x_3}{2} \right) + f\left( \frac{x_3 + x_4}{2} \right) \right)
\]

\[
= \Delta x \sum_{i=0}^{3} f\left( \frac{x_i + x_{i+1}}{2} \right)
\]

This same sequence of steps can be generalized for midpoint approximation of the definite integral \( \int_a^b f(x) \, dx \) using \( n \) subdivisions:

\[
\int_a^b f(x) \, dx \approx \left( \Delta x \cdot f\left( \frac{x_0 + x_1}{2} \right) \right) + \left( \Delta x \cdot f\left( \frac{x_1 + x_2}{2} \right) \right) + \cdots \\
+ \left( \Delta x \cdot f\left( \frac{x_n + x_{n+1}}{2} \right) \right)
\]

\[
= \Delta x \left( f\left( \frac{x_0 + x_1}{2} \right) + f\left( \frac{x_1 + x_2}{2} \right) + \cdots + f\left( \frac{x_{n-1} + x_n}{2} \right) + f\left( \frac{x_n + x_{n+1}}{2} \right) \right)
\]

\[
= \Delta x \sum_{i=0}^{n-1} f\left( \frac{x_i + x_{i+1}}{2} \right)
\]
**Trapezoid Approximation**

This time we create a trapezoid within each subdivision. The height of each trapezoid is the length of the subdivision. The two bases of each trapezoid correspond to the values of the function at the endpoints of the subinterval on which the trapezoid has been drawn.

\[ y = f(x) = e^{-x^2} \]

It may be useful to remove the first of these trapezoids and rotate it into a more conventional orientation as we calculate its area.

The sum of the areas of these four trapezoids represents an approximation for the area under the curve and therefore is one more approximation for the value of the definite integral:

\[
\int_0^1 e^{-x^2} \, dx \approx \left( \frac{1}{2} \Delta x \cdot \left( f(x_0) + f(x_1) \right) \right) + \left( \frac{1}{2} \Delta x \cdot \left( f(x_1) + f(x_2) \right) \right) + \left( \frac{1}{2} \Delta x \cdot \left( f(x_2) + f(x_3) \right) \right) + \left( \frac{1}{2} \Delta x \cdot \left( f(x_3) + f(x_4) \right) \right)
\]

\[
\approx \frac{1}{2} \Delta x \left( f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4) \right)
\]

\[
\approx \frac{1}{2} \Delta x \sum_{i=0}^{3} \left( f(x_i) + f(x_{i+1}) \right)
\]
This same sequence of steps can be generalized for trapezoid approximation of the definite integral \( \int_{a}^{b} f(x) \, dx \) using \( n \) subdivisions:

\[
\int_{a}^{b} f(x) \, dx \approx \left( \frac{1}{2} \Delta x \cdot (f(x_0) + f(x_1)) \right) + \left( \frac{1}{2} \Delta x \cdot (f(x_1) + f(x_2)) \right) + \cdots
\]

\[
+ \left( \frac{1}{2} \Delta x \cdot (f(x_{n-1}) + f(x_n)) \right)
\]

\[
\approx \frac{1}{2} \Delta x \left( f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-2}) + 2f(x_{n-1}) + f(x_n) \right)
\]

\[
\approx \frac{1}{2} \Delta x \sum_{i=0}^{n-1} (f(x_i) + f(x_{i+1}))
\]

**Simpson’s Rule for Integral Approximation**

The final approximation technique we develop in this section is called Simpson’s Rule. It is different from the first four methods because we are not creating polygons on each subinterval but rather we create a figure with a non-straight component to it. *For this method, it is required that the number of subdivisions be an even number.*

A parabola is created that contains the points \((x_0, f(x_0)), (x_1, f(x_1)), \) and \((x_2, f(x_2))\).

Another parabola is created that contains the points \((x_2, f(x_2)), (x_3, f(x_3)), \) and \((x_4, f(x_4))\).

Simpson’s Rule uses pairs of subdivisions and creates over each pair a parabola that contains the points \((x_{2i-2}, f(x_{2i-2})), (x_{2i-1}, f(x_{2i-1})), \) and \((x_{2i}, f(x_{2i}))\) for \( i \) going from 1 to \( n/2 \). A shape is created using the
resulting parabola, two vertical segments—one from \((x_{2i-2},0)\) to \((x_{2i-2}, f(x_{2i-2}))\) and one from \((x_{2i+2},0)\) to \((x_{2i+2}, f(x_{2i+2}))\)—and the segment on the x-axis with endpoints \((x_{2i-2},0)\) and \((x_{2i+2},0)\). The area of the resulting shape—such as of the red-shaded figure above or the green-shaded figure above—is calculated using the formula
\[
\Delta x \cdot \frac{1}{3} \left( f(x_{2i-2}) + 4f(x_{2i}) + f(x_{2i+2}) \right).
\]

The sum of the areas of these shapes represents an approximation for the area under the curve and therefore is an approximation for the value of the definite integral:
\[
\int_{0}^{1} e^{-x^2} \, dx \approx \left( \Delta x \cdot \frac{1}{3} \left( f(x_0) + 4f(x_1) + f(x_2) \right) \right) + \left( \Delta x \cdot \frac{1}{3} \left( f(x_2) + 4f(x_3) + f(x_4) \right) \right)
\]

This same sequence of steps can be generalized for the Simpson’s Rule approximation of the definite integral \( \int_{a}^{b} f(x) \, dx \) using \( n \) subdivisions:
\[
\int_{a}^{b} f(x) \, dx \approx \left( \Delta x \cdot \frac{1}{3} \left( f(x_0) + 4f(x_1) + f(x_2) \right) \right) + \cdots
\]
\[
+ \left( \Delta x \cdot \frac{1}{3} \left( f(x_{n-2}) + 4f(x_{n-1}) + f(x_n) \right) \right)
\]
\[
= \Delta x \cdot \frac{1}{3} \left( f(x_0) + 4f(x_1) + f(x_2) + \cdots + f(x_{n-2}) + 4f(x_{n-1}) + f(x_n) \right)
\]
\[
= \Delta x \sum_{i=1}^{n} \left( f(x_{2i-2}) + 4f(x_{2i-1}) + f(x_{2i}) \right)
\]