USING TRIG SUBSTITUTIONS

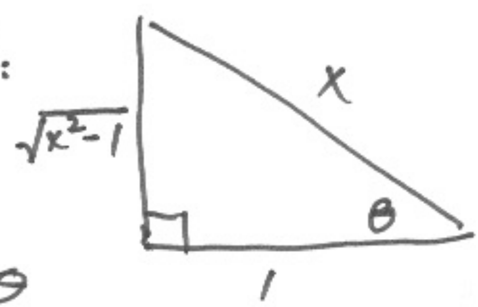
(A) $$\int \frac{3\,dx}{x^2\sqrt{x^2-1}}$$ let $x^2-1 = \sec^2\theta - 1$

$\Rightarrow x = \sec\theta$ & $dx = \sec\theta\tan\theta\,d\theta$

$$= \int \frac{3(\sec\theta\tan\theta\,d\theta)}{(\sec^2\theta)\sqrt{\sec^2\theta-1}} = 3\int \frac{\sec\theta\tan\theta\,d\theta}{\sec^2\theta\sqrt{\tan^2\theta}}$$

$$= 3\int \frac{\sec\theta\tan\theta\,d\theta}{\sec^2\theta\cdot\tan\theta} = 3\int\frac{d\theta}{\sec\theta} = 3\int\cos\theta\,d\theta$$

$$= 3\sin\theta + C$$ and:



$x = \sec\theta$

$\Rightarrow \frac{1}{x} = \cos\theta$

so $\sin\theta = \frac{\sqrt{x^2-1}}{x}$

$$= 3\left(\frac{\sqrt{x^2-1}}{x}\right) + C$$

3 $\displaystyle\int \frac{5\,dx}{\sqrt{x^2+16}}$

if $\tan^2\theta + 1 = \sec^2\theta$

$16\tan^2\theta + 16 = 16\sec^2\theta$

and $x^2 + 16 = 16\tan^2\theta + 16$

$\Rightarrow x^2 = 16\tan^2\theta$

$\Rightarrow x = 4\tan\theta$

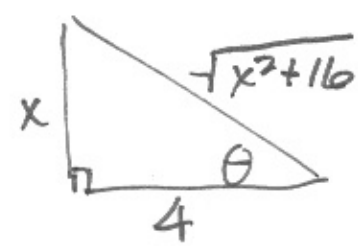
and $dx = 4\sec^2\theta\,d\theta$

$$= \int \frac{5(4\sec^2\theta\,d\theta)}{\sqrt{16\tan^2\theta+16}}$$

$$= 20\int \frac{\sec^2\theta\,d\theta}{\sqrt{16\sec^2\theta}}$$

$x = 4\tan\theta \Rightarrow \frac{x}{4} = \tan\theta$

$\Rightarrow$



$\Rightarrow \sec\theta = \frac{\sqrt{x^2+16}}{4}$

$$= 20\int \frac{\sec^2\theta\,d\theta}{4\sec\theta}$$

$$= 5\int \sec\theta\,d\theta = 5\ln\left|\sec\theta + \tan\theta\right| + C$$

$$= 5\ln\left|\frac{\sqrt{x^2+16}}{4} + \frac{x}{4}\right| + C$$

$$= 5\ln\left|\frac{\sqrt{x^2+16}+x}{4}\right| + C$$

(C) $\int_0^6 5t^2\sqrt{36-t^2}\,dt$ $\quad 1-\sin^2\theta = \cos^2\theta$

so $\quad 36-36\sin^2\theta = 36\cos^2\theta$

and if $\quad 36-t^2 = 36-36\sin^2\theta$, we have

$t^2 = 36\sin^2\theta \Rightarrow t = 6\sin\theta$

& $dt = 6\cos\theta\, d\theta$

and, looking at integration limits, if

$t=0$, we need $6\sin\theta = 0 \Rightarrow \theta = 0$; if

$t=6$, we need $6\sin\theta = 6 \Rightarrow \theta = \pi/2$, so we now have

$$\int_0^{\pi/2} 5(36\sin^2\theta)\sqrt{36\cos^2\theta}\cdot(6\cos\theta\, d\theta)$$

$$= 6480\int_0^{\pi/2} \sin^2\theta\cos^2\theta\, d\theta$$

$$= 6480\int_0^{\pi/2}\left(1-\frac{\cos(2\theta)}{2}\right)\left(1+\frac{\cos(2\theta)}{2}\right)d\theta = \frac{6480}{4}\int_0^{\pi/2} 1-\cos^2(2\theta)\, d\theta$$

$$= 1620\int_0^{\pi/2} 1-\left(\frac{1+\cos(4\theta)}{2}\right)d\theta = 810\int_0^{\pi/2} 2-1-\cos(4\theta)\, d\theta$$

$$= 810\int_0^{\pi/2} 1-\cos(4\theta)\, d\theta = 810\left(\theta - \frac{1}{4}\sin(4\theta)\right)\Big]_0^{\pi/2}$$

$$= 810\left[\left(\pi/2 - \tfrac{1}{4}(0)\right) - \left(0 - \tfrac{1}{4}(0)\right)\right] = 810\cdot\pi/2 = 405\cdot\pi$$

(D)

$$\int \frac{2x\,dx}{\sqrt{x^2-8}}$$

$$\sec^2\theta - 1 = \tan^2\theta$$

$$8\sec^2\theta - 8 = 8\tan^2\theta$$

$$8\sec^2\theta - 8 = x^2 - 8$$

$$\Rightarrow 8\sec^2\theta = x^2$$

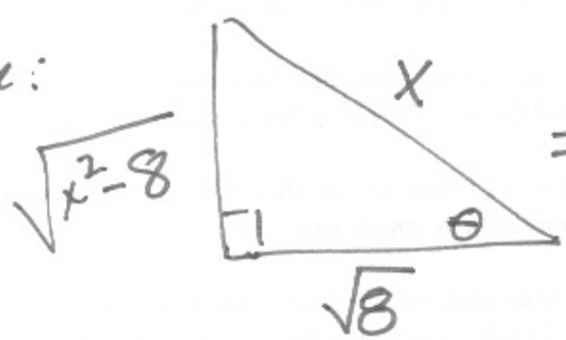
$$\Rightarrow x = \sqrt{8}\sec\theta$$

$$\text{\& } dx = \sqrt{8}\sec\theta\tan\theta\,d\theta$$

$$= \int \frac{2(\sqrt{8}\sec\theta)(\sqrt{8}\sec\theta\tan\theta)\,d\theta}{\sqrt{8\tan^2\theta}}$$

$$= 16\int \frac{\tan\theta\sec^2\theta\,d\theta}{\sqrt{8}\tan\theta} = \frac{16}{\sqrt{8}}\int \sec^2\theta = 2\sqrt{8}\tan\theta + C$$

and, with $x = \sqrt{8}\sec\theta \Rightarrow \frac{x}{\sqrt{8}} = \sec\theta$, we have:



$$\Rightarrow \tan\theta = \frac{\sqrt{x^2-8}}{\sqrt{8}}$$

so $$2\sqrt{8}\tan\theta + C = 2\sqrt{8}\cdot\frac{\sqrt{x^2-8}}{\sqrt{8}} + C$$

$$= 2\sqrt{x^2-8} + C$$