Solution Guide - Quiz #10 Samples

⑥ For $n \geq 2$, $b_n = \frac{n^2+3}{n^3-1}$ looks like $a_n = \frac{1}{n}$. We know $\sum_{n=2}^{\infty} a_n$ diverges, so we'll compare b_n to a_n. We want to show $b_n \geq a_n$. This means that $\frac{n^2+3}{n^3-1} \geq \frac{1}{n}$, or $\frac{n^2+3}{n^3-1} - \frac{1}{n} \geq 0$. So we must show that

$\frac{n^2+3}{n^3-1} - \frac{1}{n}$ is positive; $\frac{n^2+3}{n^3-1} - \frac{1}{n} = \frac{3n+1}{n^4-n} = \frac{3n+1}{n(n^3-1)}$. For $3n+1$, and $n \geq 2$, $3n+1 > 0$; For $n \geq 2$, $n > 0$ & $n^3-1 > 0$, so $n(n^3-1) > 0$. This shows $\frac{3n+1}{n(n^3-1)} > 0$, so we have $\frac{n^2+3}{n^3-1} > \frac{1}{n}$ for $n \geq 2$.

We therefore have shown that $b_n \geq a_n$, and we know $\sum_{n=2}^{\infty} a_n = \sum_{n=2}^{\infty} \frac{1}{n}$ diverges, so by the Comparison Test, $\sum \frac{n^2+3}{n^3-1}$ must DIVERGE.

⑦ Alfalfa is WRONG! $a_n = \frac{(-1)^n}{n}$ is an alternating series, with $a_n = (-1)^n \cdot b_n$, with $b_n = \frac{1}{n}$. We know that $b_{n+1} < b_n$ and we know $\lim_{n \to \infty} \frac{1}{n} = 0$; so, by the ALTERNATING SERIES TEST, with $\frac{1}{n+1} < \frac{1}{n}$ for $n \geq 1$ and $\lim_{n \to \infty} \frac{1}{n} = 0$, we know $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ CONVERGES.

* (Note that $\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n}$ is NOT CONVERGENT, so the original series, $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ is only conditionally convergent, not absolutely convergent.)

(8) We know that $\sum_{n=1}^{\infty} \frac{5^n}{4^{n+2}} = \frac{1}{16}\sum_{n=1}^{\infty}\left(\frac{5}{4}\right)^n$. This is a geometric series with $r = 5/4$. Because $\left|\frac{5}{4}\right| > 1$, the series DIVERGES.

(9) We'll use the LIMIT COMPARISON TEST here, comparing $a_n = \frac{4n^2+n}{\sqrt[3]{n^7+n^3}}$ to $b_n = \frac{1}{n^{1/3}}$ because the dominant terms in a_n numerator and denominator reduce to $\frac{1}{n^{1/3}}$.
So we look at $\lim_{n\to\infty} a_n/b_n = 4$ (TI-89 evaluation).
Therefore, because a finite positive limit exists, and because $b_n = \frac{1}{n^{1/3}}$ diverges by the p-series test, we know $\sum_{n=1}^{\infty} a_n$ diverges by the LIMIT COMPARISON TEST.

(10) The associated function is $f(x) = \frac{2x-3}{6x^2+1}$. For $x > 3/2$, we know this is positive; it is continuous, because there are no values of x, $x > 3/2$, for which $f(x)$ is not defined. Finally, looking at $f'(x) = \frac{-2(6x^2-18x-1)}{(6x^2+1)^2}$, we know $f'(x) < 0$ if $6x^2-18x-1 > 0$, and $6x^2-18x-1 = 0$ if $x_1 = -1/6(\sqrt{87}-9)$ OR $x_2 = \frac{1}{6}(\sqrt{87}+9)$. $x_2 \geq 1$, and for all $x > x_2$, $6x^2-18x-1 > 0$. This shows that for all $x > \frac{1}{6}(\sqrt{87}+9)$, $f'(x) < 0$ &, because $f'(0)$ for all $x > 1/6(\sqrt{87}+9)$, f is decreasing on $x > \frac{1}{6}(\sqrt{87}+9)$.
This shows that $f(x)$ is positive, continuous, & decreasing for $x > K$, $K = 1/6(\sqrt{87}+9)$