Non-Linear Renewal Theory with Stationary Perturbations

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ABSTRACT

A non-linear renewal theorem is obtained from random walks that are perturbed by an approximately stationary sequence. As corollaries, the limiting joint distribution of the excess over the boundary and last perturbation are obtained along with an approximation to expected first passage times. The results are illustrated by an analysis of a sequential probability ratio test when data are subject to both censoring and staggered entry.

Key Words and Phrases: asymptotic distributions, censoring, likelihood function, random walks, uniform integrability.

INTRODUCTION

Sequential problems that involve both censoring and staggered entry lead to processes that are random walks perturbed, not by slowly changing terms, but (approximately) stationary ones. To see how, consider the following simple example. Suppose that patients arrive for treatment at times $0 = \tau_0 < \tau_1 < \tau_2 < \cdots$, are treated, and live for exponentially distributed periods $Y_1, Y_2, \cdots$ thereafter, with an unknown failure rate θ. Then the data available at time τ_n consist of

$$Y_{nk} = \min[Y_k, \tau_n - \tau_{k-1}],$$
$$\Delta_{nk} = \mathbf{1}_{\{Y_k \le \tau_n - \tau_{k-1}\}}$$

for $k = 1, \cdots, n$. The log-likelihood function given $Y_{n1}, \cdots, Y_{nn}, \Delta_{n1}, \cdots, \Delta_{nn}$ is

$$\ell_n(\theta) = K_n \log(\theta) - \theta T_n,$$

where $K_n = \Delta_{n1} + \cdots + \Delta_{nn}$ and $T_n = Y_{n1} + \cdots + Y_{nn}$ are the number of failures and the total time on test at time τ_n. The sequential probability ratio test for testing $H_0 : \theta = 1$ vs $H_1 : \theta = \delta \neq 1$ may be described as follows: Let

$$\Lambda_n = \ell_n(\delta) - \ell_n(1),$$
$$N_a = \inf\{n \ge 1 : \Lambda_n > a\},$$

and

$$M_b = \inf\{n \ge 1 : \Lambda_n < -b\}$$

for $a, b \geq 0$. The test takes a sample of size $\min(N_a, M_b)$ and rejects H_0 in favor of H_1 iff $N_a < M_b$. The dominant term in the type I error probability is then

$$P_1[N_a < \infty] = \int_{\{N_a<\infty\}} e^{-\Lambda_{N_a}} dP_\delta = e^{-a} E_\delta\big[e^{-(\Lambda_{N_a}-a)}\big],$$

and the asymptotic distribution of $\Lambda_{N_a} - a$ is of interest. Following the procedure in Lai and Siegmund (1977,1979), it seem natural to write Λ_n as a perturbed random walk. This is not difficult, since

$$\Lambda_n = n\log(\delta) - (\delta-1)(Y_1 + \cdots + Y_n) + \tilde{\xi}_n,$$

where

$$\tilde{\xi}_n = (\delta-1)\sum_{j=1}^{n}[Y_{n-j+1} - (\tau_n - \tau_{n-j})]^+ - \log(\delta)\sum_{j=1}^{n}\mathbf{1}_{\{Y_{n-j+1}>\tau_n-\tau_{n-j}\}},$$

$\mathbf{1}_B$ denotes the indicator function of B, and $x^+ = \max(x, 0)$. Unfortunately, $\tilde{\xi}_n$ is not slowly changing, as required by Lai and Siegmund. However, if the arrival times τ_k form a renewal process (so that $\tau_k - \tau_{k-1},\ k = 0, \pm1, \pm2, \cdots$ are i.i.d.), then $\tilde{\xi}_n$ does have some structure, since

$$\tilde{\xi}_n \approx (\delta-1)\sum_{j=1}^{\infty}[Y_{n-j+1} - (\tau_n - \tau_{n-j})]^+ - \log(\delta)\sum_{j=1}^{\infty}\mathbf{1}_{\{Y_{n-j+1}>\tau_n-\tau_{n-j}\}} = \xi_n,$$

say, and the right side is a stationary process.

The preceding is intended to motivate the study of renewal theory for processes of the form

$$Z_n = S_n + \xi_n, \tag{1}$$

where S_n is a random walk with a positive drift μ, say, and ξ_n is a stationary sequence with common marginal distribution function G, say. The goal of the paper is to contribute to such a study. A main result is that

$$\lim_{a\to\infty}\sum_{n=1}^{\infty} P[\xi_n \leq c,\ a < Z_n \leq a+b] = \frac{b}{\mu}G(c) \tag{2}$$

for $0 < b < \infty$ and continuity points c of G, under modest conditions. Denote the first passage times and excesses by

$$t_a = \inf\{n \geq 1 : Z_n > a\} \tag{3}$$

$$R_a = Z_{t_a} - a. \tag{4}$$

Then the asymptotic joint distribution of ξ_{t_a} and R_a may be obtained as from (2). It is shown that ξ_{t_a} and R_a have asymptotic joint distribution function,

$$H(c,r) = \frac{1}{\mu}\int_0^r P[\xi_0 \le c, M \ge u]du \tag{5}$$

where

$$M = \inf_{k<0} Z_k$$

and

$$Z_k = X_{k+1} + \cdots + X_0 + (\xi_0 - \xi_k)$$

for $k \le -1$. Under additional moment conditions, it is shown that R_a and ξ_{t_a} are uniformly integrable in $a > 0$ and that the main results continue to holds for approximately stationary sequences. The main results are specialized to the exponential case in Section 2, and the accuracy of the resulting approximations is assessed using simulation. Formal statements and proofs of the main results are presented in Section 4 and 5. Section 3 contains some preliminary lemmas, and Section 6 some concluding remarks.

The asymptotic (marginal) distribution of R_a is not new. It may be obtained from Lalley's (1986) general renewal theorem for sums of stationary processes. Relation (2), (5), and the uniform integrability are new, however (to the best of the authors' knowledge), and the derivation of the asymptotic distribution differs greatly from Lalley's. The work of Melfi (1992,1995) and Su (1993,1998) is related.