Testing for a Change in the Hazard Rate with Staggered Entry

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Dedicated to Professor Z. Govindarajulu on his 70th birthday

ABSTRACT

We develop a likelihood-ratio based procedure for testing for a change in the hazard rate of the patients' survival distribution when the patients enter the trial at random times. We show that the profile log-likelihood ratio process converges weakly to a non-stationary Gaussian process and obtain the upper percentiles of the supremum of the limiting process using an approximation for the tail distribution of the supremum of the Ornstein–Uhlenbeck process. We also discuss the power of the test.

Key Words and Phrases: change-point; likelihood ratio test; Gaussian processes; Monte Carlo simulation; Poisson process.

1. INTRODUCTION

Suppose that the patients arrive for treatment at times $0 < \tau_1 < \tau_2 < \cdots$, following a Poisson process with rate γ. Let N be the total number of patients who arrived in the time interval $[0, T]$. For $i = 1, \ldots, N$, let Y_i be the survival time of the ith patient; and suppose that the distribution of the Y_i is of the form

$$f_Y(y) = \begin{cases} \lambda_1 e^{-\lambda_1 y} & \text{if } y < \nu; \\ \lambda_2 e^{-\lambda_1 \nu - \lambda_2 (y-\nu)} & \text{if } y \geq \nu, \end{cases} \tag{1}$$

where $0 < \lambda_1,\ \lambda_2,\ \nu$ are unknown parameters. That is, the failure rate may change at an unknown time ν. The model is irregular in that ν only has meaning if $\lambda_1 \neq \lambda_2$.

In this paper we develop a procedure for testing the hypotheses $H_0 : \lambda_1 = \lambda_2$ v.s. $H_1 : \lambda_1 \neq \lambda_2$ based on the observations

$$\begin{aligned} X_i &= \min(Y_i, T - \tau_i), \\ \delta_i &= \mathbf{1}_{\{Y_i \leq T - \tau_i\}}, \end{aligned}$$

where $\mathbf{1}_A$ denotes the indicator of an event A. The log-likelihood function for this data is

$$\ell_N\left[\lambda_1, \lambda_2, \nu\right] = K_1 \log \lambda_1 - \lambda_1 T_1 + K_2 \log \lambda_2 - \lambda_2 T_2$$

where

$$\begin{aligned} K_1 &= K_1(\nu) = \sum_{i=1}^{N} 1\{X_i < \nu,\, \delta_i = 1\}, \\ K_2 &= K_2(\nu) = \sum_{i=1}^{N} 1\{X_i \geq \nu,\, \delta_i = 1\}, \\ T_1 &= T_1(\nu) = \sum_{i=1}^{N} (X_i \wedge \nu), \\ T_2 &= T_2(\nu) = \sum_{i=1}^{N} (X_i - \nu)^+, \end{aligned} \tag{2}$$

$x \wedge \nu$ denotes the smaller of x and ν, and $x^+ = \max(x, 0)$.

For given ν, the maximum likelihood estimators (MLE) of λ_1 and λ_2 are $\hat{\lambda}_1(\nu) = K_1(\nu)/T_1(\nu)$ and $\hat{\lambda}_2(\nu) = K_2(\nu)/T_2(\nu)$, respectively. Under the null hypothesis the MLE of the common rate λ is $\hat{\lambda} = (K_1 + K_2)/(T_1 + T_2)$, which does not depend on ν. The profile log-likelihood ratio statistic for testing H_0 versus H_1 for a fixed ν is then

$$\begin{aligned} \Lambda_N(\nu) &= \ell_N\left[\hat{\lambda}_1(\nu), \hat{\lambda}_2(\nu), \nu\right] - \ell_N(\hat{\lambda}, \hat{\lambda}) \\ &= K_1 \log\left[\frac{\hat{\lambda}_1(\nu)}{\hat{\lambda}}\right] + K_2 \log\left[\frac{\hat{\lambda}_2(\nu)}{\hat{\lambda}}\right], \end{aligned}$$

where the conventions $0 \cdot \infty = 0 = 0/0$ are observed.

The test that we study depends on two design parameters $0 < a < b < 1$ and rejects H_0 if $\sup_{aT \leq \nu \leq bT} \Lambda_N(\nu)$ exceeds a critical value that is identified below. The inclusion of a and b here is necessary because a change at $\nu = 0$ or $\nu = T$ is indistinguishable from H_0 within the context of the model. Alternatively, the test only looks for changes that occur between aT and bT.

Several authors have studied tests for a change point in the hazard rate using the likelihood ratio statistic. Matthews and Farewell (1982) considered the problem in a different formulation of the model based on uncensored observations. Based on a simulation study they suggested that moderate amount of Type I censoring has little impact on the null distribution of the test statistic. Worsley (1988) obtained the exact null distribution of a restricted version of the likelihood ratio test statistic. He showed that under the null hypothesis the exact distribution of test statistic remains unchanged for Type II censoring, but it heavily depends on the interval to which the change point is assumed to belong. Loader (1991) considered the problem and, using a heuristic argument, concluded that the effect of random censoring on the significance level to be relatively minor. Matthews et al. (1985) studied the hazard rate change using the normalized score statistic and showed that the score process weakly converges to an Ornstein–Uhlenbeck process when λ is known and to a Brownian bridge when λ is unknown. Our approach differs from these studies by its focus on staggered entry, where type I censoring is naturally built into the model.