Topic: Integration by parts

Background:

$$\int udv = uv - \int vdu$$

Choose u, dv such that $\int dv$ and $\int vdu$ are easier than the original $\int udv$

In some cases $\int vdu$ is not readily computable although it is simpler than (or similar to) the original integral $\int udv$. In such cases, applying integration by parts to $\int vdu$ often does the trick. Examples (3) and (6) below illustrate this idea.

Illustrative Examples:

(1) Find the following indefinite integral .

$$\int x\sin(x)dx$$

Solution:

Choose $u = x, dv = \sin(x)dx$. Thus, $v = \int \sin(x)dx = -\cos(x), du = dx$.

Hence,

$$\begin{aligned}\int x\sin(x)dx &= x(-\cos(x)) - \int -\cos(x)dx \\ &= -x\cos(x) + \sin(x) + C\end{aligned}$$

(2) Find the following indefinite integral .

$$\int xe^x dx$$

Solution:

Choose $u = x, dv = e^x dx$. Thus, $v = \int e^x dx = e^x, du = dx$.

Hence,

$$\begin{aligned}\int xe^x dx &= xe^x - \int e^x dx \\ &= xe^x - e^x + C\end{aligned}$$

(3) Find the following indefinite integral .

$$\int x^2 e^x dx$$

Solution:

Choose $u = x^2, dv = e^x dx$. Thus, $v = \int e^x dx = e^x, du = 2xdx$.

Hence,

$$\begin{aligned} \int x^2 e^x dx &= x^2 e^x - \int 2xe^x dx \\ &= x^2 e^x - 2(xe^x - e^x) + C \quad \text{(using the result in Problem (2))} \\ &= x^2 e^x - 2xe^x + 2e^x + C \end{aligned}$$

(4) Compute the following definite integral.

$$\int_0^1 xe^{-x} dx$$

Solution:

We first compute the indefinite integral.

Choose $u = x, dv = e^{-x} dx$. Thus, $v = \int e^{-x} dx = -e^{-x}, du = dx$.

Hence,

$$\begin{aligned} \int xe^{-x} dx &= -xe^{-x} - \int -e^{-x} dx \\ &= -xe^{-x} - e^{-x} + C \end{aligned}$$

Hence,

$$\begin{aligned} \int_0^1 xe^{-x} dx &= -xe^{-x} - e^{-x}|_0^1 \\ &= (-e^{-1} - e^{-1}) - (0 - e^0) \\ &= -\frac{2}{e} + 1 \end{aligned}$$

(5) Find the following indefinite integral .

$$\int \ln(x) dx$$

<u>Solution</u>:

Choose $u = \ln(x), dv = dx$. Thus, $v = \int dx = x, du = \frac{1}{x}$.

Hence,

$$\begin{aligned} \int \ln(x)dx &= x\ln(x) - \int x\frac{1}{x}dx \\ &= x\ln(x) + x + C \end{aligned}$$

(6) Find the following indefinite integral .

$$\int e^x \cos(x)dx$$

<u>Solution</u>:

Choose $u = e^x, dv = \cos(x)dx$. Thus, $v = \int \cos(x)dx = \sin(x), du = e^x dx$.

Hence,

$$\int e^x \cos(x)dx = e^x \sin(x) - \int e^x \sin(x)dx \qquad (1)$$

Let us apply integration by parts to $\int e^x \sin(x)dx$.

Choose $u = e^x, dv = \sin(x)dx$. Thus, $v = \int \sin(x)dx = -\cos(x), du = e^x dx$

Hence,

$$\begin{aligned} \int e^x \sin(x)dx &= -e^x \cos(x) - \int -e^x \cos(x)dx \\ &= -e^x \cos(x) + \int e^x \cos(x)dx \qquad (2) \end{aligned}$$

Substituting equation (2) into (1) we have,

$$\begin{aligned} \int e^x \cos(x)dx &= e^x \sin(x) - (-e^x \cos(x) + \int e^x \cos(x)dx) \\ &= e^x \sin(x) + e^x \cos(x) - \int e^x \cos(x)dx \end{aligned}$$

Now solving for $\int e^x \cos(x)dx$ we have,

$$\int e^x \cos(x)dx = \frac{e^x \sin(x) + e^x \cos(x)}{2}$$

.

Cautions:

$$\boxed{\int f(x)g(x)dx \neq (\int f(x)dx)(\int g(x)dx)}$$

Example:

$$\int x\sin(x)dx \neq (\int xdx)(\int \sin(x)dx) = -\frac{x^2\cos(x)}{2}$$

Illustrative example (1) shows the correct way to compute $\int x\sin(x)dx$.

In illustrative example (1) we obtained

$$\int x\sin(x)dx = -x\cos(x) + \sin(x) + C$$

$$\boxed{\int \left(\frac{f(x)}{g(x)}\right) dx \neq \frac{\int f(x)dx}{\int g(x)dx}}$$

Example:

$$\int \frac{x}{e^x}dx \neq \frac{\int xdx}{\int e^x dx} = \frac{1}{e^x}$$

Illustrative example (4) shows the correct way to compute $\int \frac{x}{e^x}dx$.

In illustrative example (4) we obtained

$$\int \frac{x}{e^x}dx = -xe^{-x} - e^{-x} + C$$