Topic: Quadratic functions

Background:

Real solutions to $ax^2 + bx + c = 0$ are:

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ and } x = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \text{ provided } b^2 - 4ac \geq 0$$

Note: The number $b^2 - 4ac$ is called the *discriminant* of the equation $ax^2 + bx + c = 0$.

If the discriminant is negative then the equation has no real solutions.

If $\sqrt{b^2 - 4ac} \geq 0$, then $ax^2 + bx + c$ factors into $a(x - u)(x - v)$, where

$$u = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ and } v = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

Easy case: If $\sqrt{b^2 - 4c} \geq 0$, then $x^2 + bx + c$ factors into $(x - u)(x - v)$, where u,v satisfy:

$$uv = c \text{ and } u + v = b$$

Completing the square: $ax^2 + bx + c = a(x - m)^2 + n$, where

$$m = -\frac{b}{2a} \text{ and } n = c - \frac{b^2}{4a}$$

Illustrative Examples:

(1) Factorize $3e^{2t} - 7e^t + 2$.

Solution:

Let $x = e^t$. Then we have,

$3e^{2t} - 7e^t + 2 = 3x^2 - 7x + 2 = 3x^2 - 6x - x + 2 = 3x(x-2) - (x-2) = (x-2)(3x-1)$
$= (e^t - 2)(3e^t - 1)$.

(2) Solve $3x^2 - 8x + 2 = 0$ by completing the square and by the quadratic formula.

Solution:

Using the formula for completing the square with $a = 3, b = -8, c = 2$ we have,

$$m = -\frac{(-8)}{2(3)} = \frac{4}{3}, n = 2 - \frac{(-8)^2}{4(3)} = -\frac{10}{3}$$

$$\text{Hence, } 3x^2 - 8x + 2 = 0 \quad \Rightarrow \quad 3\left(x - \frac{4}{3}\right)^2 - \frac{10}{3} = 0.$$

$$\Rightarrow \quad 3\left(x - \frac{4}{3}\right)^2 = \frac{10}{3}.$$

$$\Rightarrow \quad \left(x - \frac{4}{3}\right)^2 = \frac{10}{9}.$$

$$\Rightarrow \quad x = \frac{\sqrt{10}}{3} + \frac{4}{3} \text{ or } x = -\frac{\sqrt{10}}{3} + \frac{4}{3}.$$

Alternately, using the quadratic formula with $a = 3, b = -8, c = 2$ we have,

$$x = \frac{-(-8) + \sqrt{(-8)^2 - 4(3)(2)}}{2(3)}, \text{ or, } x = \frac{-(-8) - \sqrt{(-8)^2 - 4(3)(2)}}{2(3)}$$

$$\text{i.e. } x = \frac{8 + \sqrt{40}}{6}, \text{ or, } x = \frac{8 - \sqrt{40}}{6}$$

$$x = \frac{4}{3} + \frac{\sqrt{10}}{3} \text{ or } x = \frac{4}{3} - \frac{\sqrt{10}}{3}$$

(3) Express $4x^2 - 16x + 2$ in the form $4(x-h)^2 + k$ for some constants h and k.

Solution:

$4x^2 - 32x + 2 = 4(x^2 - 8x) + 2 = 4(x^2 - 8x + 16) - 64 + 2 = 4(x-4)^2 - 62$.

(4) Find all solutions to $4(t+7)^2 - 32(t+7) + 2 = 0$.

Solution:

Let $x = (t+7)$. Then,

$$\begin{aligned}
4(t+7)^2 - 32(t+7) + 2 = 0 \quad &\Rightarrow \quad 4x^2 - 32x + 2 = 0. \\
&\Rightarrow \quad x = \frac{32 + \sqrt{(32)^2 - 32}}{8} \text{ or } x = \frac{32 - \sqrt{(32)^2 - 32}}{8}. \\
&\Rightarrow \quad x = 4 + \sqrt{\frac{31}{2}} \text{ or } x = 4 - \sqrt{\frac{31}{2}}. \\
&\Rightarrow \quad t = 4 + \sqrt{\frac{31}{2}} - 7 \text{ or } t = 4 - \sqrt{\frac{31}{2}} - 7. \\
&\Rightarrow \quad t = -3 + \sqrt{\frac{31}{2}} \text{ or } t = -3 - \sqrt{\frac{31}{2}}.
\end{aligned}$$

(5) Find the discriminant of the quadratic equation $4x^2 - 5x + 2 = 0$ and all real solutions to this equation.

Solution:

The discriminant of the quadratic equation $ax^2 + bx + c = 0$ is $b^2 - 4ac$.

Hence, the discriminant of the given equation is $(-5)^2 - 4(4)(2) = -7 < 0$ which in turn implies that the equation has no real solutions.