Math 146
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11.2: Telescoping Series and the Test for Divergence

We have already seen that if $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$. Taking the contrapositive of this statement, we have the following:

Test for Divergence: If $\lim_{n\to\infty} a_n \neq 0$ or $\lim_{n\to\infty} a_n$ does not exist, then $\sum_{n=1}^{\infty} a_n$ diverges.

(1) Is it true that if $\lim_{n\to\infty} a_n = 0$, then $\sum_{n=1}^{\infty} a_n$ converges? If so, prove it. Otherwise, give an example of a divergent series $\sum_{n=1}^{\infty} a_n$ with $\lim_{n\to\infty} a_n = 0$.

(2) Let $a_n = \frac{2n}{3n+1}$.

(a) Is $\{a_n\}$ convergent? Explain.

(b) Is $\sum_{n=1}^{\infty} a_n$ convergent? Explain.

In general, it is very hard to find the sum of a convergent series. So far, you can only find the sum of a **geometric** series. Recall that $\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$ if $|r| < 1$ and that this series diverges otherwise. Another type of series for which we can find its sum is called a **telescoping** series. The series $\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right)$, from last class period, is one such example. Let's consider a few more examples.

(3) Consider the series $\sum_{n=1}^{\infty} \frac{2}{n^2+4n+3}$.

(a) Express $\frac{2}{n^2+4n+3}$ as a sum of two fractions (i.e., use the technique of "partial fractions").

(b) Find an expression for s_n, the nth partial sum.

(c) Determine whether or not the series is convergent or divergent.

(4) Determine whether $\sum_{n=1}^{\infty} \ln\left(\frac{n}{n+1}\right)$ is convergent or divergent by finding an expression for s_n and then finding $\lim_{n\to\infty} s_n$.

Another useful fact is the following:

Theorem 1 *If* $\sum a_n$ *and* $\sum b_n$ *are two convergent series, then*

$$\sum ca_n$$

(c is a constant) and

$$\sum (a_n + b_n)$$

are convergent series. Furthermore,

$$\sum ca_n = c \sum a_n$$

and

$$\sum (a_n + b_n) = \sum a_n + \sum b_n.$$

(5) Answer each of the following **TRUE** or **FALSE**. Justify your answers by providing either an example to show a statement is false or a few sentences describing why a statement is true.

(a) If $c, a_n > 0$ and $\sum a_n$ diverges, then $\sum ca_n$ diverges.

(b) If $\sum (a_n + b_n)$ converges, then each of the series $\sum a_n$ and $\sum b_n$ converge.

(c) If $a_n, b_n > 0$ and $\sum b_n$ diverges, then $\sum (a_n + b_n)$ diverges.

(d) If $\sum b_n$ diverges, then $\sum (a_n + b_n)$ diverges.

(6) Determine whether the following series are convergent or divergent. If a series is convergent, find its sum.

(a) $\displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{e^n}+\frac{1}{n(n+1)}\right)$

(b) $\displaystyle\sum_{n=1}^{\infty}\frac{1+3^n}{2^n}$

(c) $\displaystyle\sum_{n=1}^{\infty}\left(\frac{3}{5^n}+\frac{2}{n}\right)$