Math 146
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Section 7.7: Approximate Integration

Sometimes, it's impossible to find the exact value of a definite integral $\int_a^b f(x)\,dx$. It may be difficult, or even impossible, to find an antiderivative, or it may be that the function f is determined from collected data and we may not have a formula for f or be able to determine one. In these cases, we want to **approximate** the value of the definite integral using Riemann sums. In Chapter 5, you should have learned at least two such methods, **Left Endpoint Rule** and **Right Endpoint Rule**, and maybe also the **Midpoint Rule**. In this section, we will also learn the **Trapezoid Rule** and **Simpson's Rule**.

Consider the definite integral $\int_0^1 e^{-x^2}\,dx$. There is no antiderivative for this function so our only option is to approximate the value of this definite integral.

1. Draw the graph of $y = e^{-x^2}$ on $[0, 1]$ below. Use all the space provided!

2. In the graph above, divide the interval $[0, 1]$ into $n = 10$ subintervals whose endpoints are $x_1, x_2, \ldots, x_{10}$. Let Δx denote the length of one subinterval. What is the value of Δx?

3. Fill in the table below. Use approximate function values that are correct to four decimal places.

i	x_i	$f(x_i)$
0	0	
1		
2		
3		
4		
5		
6		
7		
8		
9		
10	1	

4. Now let's begin to approximate the value of $\int_0^1 e^{-x^2}\,dx$.

(a) Let's start with computing L_{10}, the Left Endpoint Rule with $n = 10$ subintervals; i.e., we divide the area under the curve of $y = e^{-x^2}$ from $a = 0$ to $b = 1$ into 10 rectangles, whose heights are determined by the **left endpoint** of each subinterval and whose bases all have length Δx. Use your table from the previous question to determine L_{10} by filling the blanks below.

$$\begin{aligned} L_{10} &= \Delta x(f(x_0) + f(\_\_\_) + f(\_\_\_) + f(\_\_\_) + f(\_\_\_) + f(\_\_\_) + f(\_\_\_) \\ &\qquad + f(\_\_\_) + f(\_\_\_) + f(\_\_\_)) \\ &= \_\_\_(\_\_\_\_\_\_ + \_\_\_\_\_\_ + \_\_\_\_\_\_ + \_\_\_\_\_\_ + \_\_\_\_\_\_ \\ &\qquad + \_\_\_\_\_\_ + \_\_\_\_\_\_ + \_\_\_\_\_\_ + \_\_\_\_\_\_ + \_\_\_\_\_\_) \\ &= \_\_\_\_\_\_\_\_ \end{aligned}$$

(b) Similarly, we can approximate the value of this definite integral using the Right Endpoint Rule with $n = 10$ subintervals, R_{10}. So, this time, when dividing the area up into 10 rectangles, the heights of the rectangles are determined by the **right endpoint** of each subinterval.

$$\begin{aligned} R_{10} &= \Delta x(f(\_\_\_) + f(\_\_\_) + f(\_\_\_) + f(\_\_\_) + f(\_\_\_) + f(\_\_\_) + f(\_\_\_) \\ &\qquad + f(\_\_\_) + f(\_\_\_) + f(x_{10})) \\ &= \_\_\_(\_\_\_\_\_\_ + \_\_\_\_\_\_ + \_\_\_\_\_\_ + \_\_\_\_\_\_ + \_\_\_\_\_\_ \\ &\qquad + \_\_\_\_\_\_ + \_\_\_\_\_\_ + \_\_\_\_\_\_ + \_\_\_\_\_\_ + \_\_\_\_\_\_) \\ &= \_\_\_\_\_\_\_\_ \end{aligned}$$

(c) We could also use the **midpoint** of each subinterval to calculate the height of our rectangle. What's the midpoint of the first subinterval? The second? The third?

Fill in the blanks below to find M_{10}, the Midpoint Rule with $n = 10$ subintervals. Again use approximate function values that are correct to four decimal places.

$$\begin{aligned} M_{10} &= \Delta x(f(.05) + f(\_\_\_) + f(\_\_\_) + f(\_\_\_) + f(\_\_\_) + f(\_\_\_) + f(\_\_\_) \\ &\qquad + f(\_\_\_) + f(\_\_\_) + f(\_\_\_)) \\ &= \_\_\_(\_\_\_\_\_\_ + \_\_\_\_\_\_ + \_\_\_\_\_\_ + \_\_\_\_\_\_ + \_\_\_\_\_\_ \\ &\qquad + \_\_\_\_\_\_ + \_\_\_\_\_\_ + \_\_\_\_\_\_ + \_\_\_\_\_\_ + \_\_\_\_\_\_) \\ &= \_\_\_\_\_\_\_\_ \end{aligned}$$

5. Now suppose that instead of dividing up the area under the curve into rectangles, we form **trapezoids** by dividing the interval $[a, b]$ into n subintervals and joining the left endpoint to the right endpoint of each subinterval with a straight line. Recall that the area of a trapezoid with base b and heights h_1 and h_2 is $\frac{b}{2}(h_1 + h_2)$. Use this information to approximate $\int_0^1 e^{-x^2}\,dx$ with the Trapezoidal Rule T_{10} with $n = 10$ by filling in the blanks below.

$$\begin{aligned}
T_{10} &= \frac{\Delta x}{2}(f(x_0) + f(x_1)) + ___(f(___) + f(___)) + ___(f(___) + f(___)) + ___(f(___) + f(___)) \\
&\quad + ___(f(___) + f(___)) + ___(f(___) + f(___)) + ___(f(___) + f(___)) \\
&\quad + ___(f(___) + f(___)) + ___(f(___) + f(___)) + \frac{\Delta x}{2}(f(x_9) + f(x_{10})) \\
&= \frac{\Delta x}{2}(f(x_0) + 2f(x_1) + ___f(x_2) + ___f(___) + ___f(___) + ___f(___) + ___f(___) \\
&\quad + ___f(___) + ___f(___) + ___f(___) + f(x_{10})) \\
&= ___(L_{10} + R_{10}) \\
&= __________
\end{aligned}$$

6. The last rule for approximation to consider is **Simpson's Rule**. The Trapezoidal Rule fits a straight line to the left and right endpoints of each subinterval. Simpson's Rule uses two subintervals at a time which gives three points on the curve; it fits a **parabola** to these points and finds the area under the parabola. For example, if we consider the first two subintervals $[x_0, x_1]$ and $[x_1, x_2]$, then it turns out that the area under the parabola passing through the points $(x_0, f(x_0)), (x_1, f(x_1)), (x_2, f(x_2))$ is

$$\frac{\Delta x}{3}[f(x_0) + 4f(x_1) + f(x_2)].$$

Use Simpson's Rule with $n = 10$ subintervals to approximate the value of $\int_0^1 e^{-x^2}\,dx$ by filling in the blanks below.

$$\begin{aligned}
S_{10} &= \frac{\Delta x}{3}(f(x_0) + 4f(x_1) + f(x_2)) + ___(f(___) + 4f(___) + f(___)) \\
&\quad + ___(f(___) + 4f(___) + f(___)) + ___(f(___) + 4f(___) + f(___)) \\
&\quad + ___(f(___) + 4f(___) + f(___)) \\
&= \frac{\Delta x}{3}[f(x_0) + 4f(x_1) + 2f(x_2) + ___f(___) + ___f(___) + ___f(___) + ___f(___) \\
&\quad + ___f(___) + ___f(___) + ___f(___) + f(x_{10})] \\
&= __________
\end{aligned}$$

7. Use your calculator to get an approximation for $\int_0^1 e^{-x^2}\,dx$. Compare the calculator's answer to the values you found for $L_{10}, R_{10}, M_{10}, T_{10}$, and S_{10} by determining the difference between what the calculator gave and the given approximation method. Which of your approximations is the best? Worst? Why do you think this is?

In general, to approximate $\int_a^b f(x)\,dx$ using n subintervals, let $\boxed{\Delta x = \frac{b-a}{n}}$ and $\boxed{x_i = a + i \cdot \Delta x.}$ Then,

- **Left:** $L_n = \Delta x[f(x_0) + f(x_1) + \cdots + f(x_{n-1})]$
- **Right:** $R_n = \Delta x[f(x_1) + f(x_2) + \cdots + f(x_n)]$
- **Midpoint:** $M_n = \Delta x[f(x_0 + .5\Delta x) + f(x_1 + .5\Delta x) + \cdots + f(x_{n-1} + .5\Delta x)]$
- **Trapezoidal:** $T_n = \frac{1}{2}[L_n + R_n]$
  $= \frac{\Delta x}{2}[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)]$
- **Simpson's:** $S_n = \frac{\Delta x}{3}[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + \cdots + 4f(x_{n-1}) + f(x_n)]$

where n must be even for Simpson's Rule.