**Math 146**
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## Section 6.5: Average Value

Suppose $f$ is a continuous function over an interval $[a, b]$, and suppose we want to calculate the **average value,** $f_{\text{avg}}$, of $f$ over the interval $[a, b]$. How do we do this?

Normally, to find an average, you sum all the values under consideration and divide by the number of values. For example, if we wanted to find the average of $76, 78, 83, 82, 91, 85$, we compute $(76 + 78 + 83 + 82 + 91 + 85)/6 = 82.5$. So, if we wanted to do the same thing to find the average value of $f$ over $[a, b]$, we would have to find the value of $f$ at every point in $[a, b]$, sum up all these values, and divide the number of values. The problem is that **there are an infinite number of values of** $f$ **in** $[a, b]$. So, how will we do this? And how does the definite integral help?

Recall that

$$\begin{aligned}\int_a^b f(x)dx &= \lim_{n\to\infty} \Delta x\ (f(x_1^*) + f(x_2^*) + \cdots + f(x_n^*)) \\ &= \lim_{n\to\infty} \frac{b-a}{n}\ (f(x_1^*) + f(x_2^*) + \cdots + f(x_n^*))\end{aligned}$$

where $x_i^*$ is any point in the interval $[x_{i-1}, x_i]$ and $x_0 = a, x_1 = a + \Delta x, x_2 = a + 2\Delta x, \ldots, x_n = b$.

So how is this going to help us compute an average value of $f$ over $[a, b]$? Well, one thing we might do is take some **sample points** at which to compute the average. For example, suppose we divide $[a, b]$ up into $n$ **segments** of equal length, pick a point $x_i^*$ in each segment, compute $f(x_i^*)$ for $i = 1, 2, \ldots, n$, sum up these $n$ values, and divide by $n$. That will give us an **approximation** to the average value! That is,

$$f_{\text{avg}} \approx \frac{f(x_1^*) + f(x_2^*) + \cdots + f(x_n^*)}{n}$$

So, if we wanted to find $f_{\text{avg}}$, what we really want to do is take more and more points, that is, let $n$ get really large. Hey, I think there's a limit in here! That is,

$$f_{\text{avg}} = \lim_{n\to\infty} \frac{f(x_1^*) + f(x_2^*) + \cdots + f(x_n^*)}{n}$$

1. Look at the expression for $\int_a^b f(x)dx$ and the expression for $f_{\text{avg}}$. How would you go from one to the other? That is, fill in the blanks below.

   $f_{\text{avg}} =$ ______ $\int_a^b f(x)dx$

   and

   $\int_a^b f(x)dx =$ ______ $f_{\text{avg}}$

2. Use your work in the previous question to find the average value, $f_{\text{avg}}$, of the function $f(x) = x^2 + 3x - 1$ on the interval $[-1, 2]$.

Not only can we compute $f_{\text{avg}}$ but it turns out that there is a value $c$ in $[a, b]$ such that $f_{\text{avg}} = f(c)$. This means that $f$ actually attains its average value on this interval! Note that this isn't always true for an average of a finite set. For example, the average of $76, 78, 83, 82, 91, 85$ is $82.5$ yet $82.5$ is **not** one of the values $76, 78, 83, 82, 91, 85$.

**Mean Value Theorem for Integrals**

Let $f$ be a continuous function on $[a, b]$. There exists a real number $c$ in $[a, b]$ such that

$$f(c) = \frac{1}{b-a} \int_a^b f(x)\, dx.$$

3. For the function in Question 2, find a real number $c$ in $[-1, 2]$ with $f(c) = f_{\text{avg}}$. Graph the function on the interval $[-1, 2]$ and draw a rectangle whose area is the same as $\int_a^b f(x)\, dx$. (Note: your rectangle should have as its base the segment $[-1, 2]$ of the $x$-axis.)

4. Let $f(x) = \sin x$ on $[0, \pi]$. Find (i) $f_{\text{avg}}$, (ii) all real numbers $c$ in $[0, \pi]$ with $f(c) = f_{\text{avg}}$, and (iii) graph the function on $[0, \pi]$ and draw a rectangle whose area is equal to $\int_a^b f(x)\, dx$.