11.4: The Comparison Tests

(1) Consider the series $\displaystyle\sum_{n=1}^{\infty} \frac{9^n}{3+10^n}$.

(a) Find a geometric series $\displaystyle\sum_{n=1}^{\infty} c^n$ with c^n "like" $\dfrac{9^n}{3+10^n}$.

(b) Does your geometric series converge or diverge?

(c) How do the terms of your geometric series compare with the terms of the original series? I.e., determine an inequality relating these two terms.

(d) What do you conclude about the convergence or divergence of the series $\displaystyle\sum_{n=1}^{\infty} \frac{9^n}{3+10^n}$?

(2) To sum up what you have just discovered, fill in the following blank: If $0 < a_n \leq b_n$ and $\displaystyle\sum_{n=1}^{\infty} b_n$ converges, then $\displaystyle\sum_{n=1}^{\infty} a_n$ ____________________.

(3) Consider the series $\sum_{n=1}^{\infty} \frac{n+1}{n\sqrt{n}}$.

(a) Find a p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ with $\frac{1}{n^p}$ "like" $\frac{n+1}{n\sqrt{n}}$.

(b) Does your p-series converge or diverge?

(c) How do the terms of your p-series compare with the terms of the original series? I.e., determine an inequality relating these two terms.

(d) What do you conclude about the convergence or divergence of the series $\sum_{n=1}^{\infty} \frac{n+1}{n\sqrt{n}}$?

(4) To sum up what you have just discovered, fill in the following blank: If $0 < a_n \leq b_n$ and $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ ____________________.

In summary, we have the following test.

Comparison Test: Let $0 < a_n \leq b_n$ (so a_n and b_n are positive).

- If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.
- If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverges.

Note that the above test **does not** give any information if $\sum b_n$ diverges or if $\sum a_n$ converges.

In general, you will need to make a **guess** about your series before applying the Comparison Test: If your "guess" is convergent, find a convergent **larger** series; if your "guess" is divergent, find a divergent **smaller** series. Good series to use for comparison are geometric series and p-series (including constant multiples of p-series).

(1) Determine, with explanation, whether the following series are convergent or divergent.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2 - 1}{3n^4 + 1}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{1 + \sin n}{10^n}$

(2) Consider the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2 + 1}}$.

(a) What p-series is this series like?

(b) How do the terms of the p-series compare with the terms in **this** series? I.e., determine an inequality relating the two.

(c) Can you use the Comparison Test in this case? Why or why not?

Thus, we need another test to handle this situation.

Limit Comparison Test: Let $a_n, b_n \geq 0$. If

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c$$

where c is a finite number and $c > 0$, then $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both converge or both diverge.
Note that the Limit Comparison Test **does not work** if $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ or $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$.

(d) Use the Limit Comparison Test with your p-series and the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+1}}$ to decide this series converges or diverges.

(3) Determine, with explanation, whether the following series are convergent or divergent.

(a) $\sum_{n=1}^{\infty} \frac{n+2}{(n+1)^3}$

(b) $\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$