Math 146
H. Jordon

Section 9.1: Differential Equations

In this chapter, we will be studying **differential equations**, equations which involve derivatives of some unknown function. The goal is to use the differential equation to recover the function or relationship between the variables involved.

Anytime you integrate a function, you are solving a differential equation. For example, what is y if $\frac{dy}{dx} = 2x$? Clearly, $y = \int 2x\ dx = x^2 + C$ is the **general solution** to this differential equation. If we want a **specific solution** to this differential equation, we also need to specify a point. That is, the **specific solution** to $\frac{dy}{dx} = 2x$ passing through the point $(0, 2)$ is $y = x^2 + 2$.

In Section 9.1, you will be introduced to differential equations **without** emphasis on solving the differential equation, while the remaining sections of this chapter will focus on solving some special types of differential equations. Of course, if our differential equation looks like $y' = f(x)$, then the equation is easy to solve (most of the time); we just integrate! However, in some situations, we cannot describe the derivative solely as a function of x, that is, for some physical situations, we have to describe the derivative as a function of x and/or y. This makes solving differential equations much more challenging.

1. Consider the differential equation $y' = 1 + y^2$. We will see how to solve this differential equation in a later section; for now, let's **verify** whether or not the following functions are solutions to this differential equation.

   (a) Is $y = (\tan x) + 3$ a solution to this differential equation?

   (b) Is $y = 3 \tan x$ a solution to this differential equation?

   (c) Is $y = \tan(x + 3)$ a solution to this differential equation?

   (d) Based on your answers to the previous questions, what do you think is the **general solution** to this differential equation?

2. Verify that every member of the family functions $y = \dfrac{3}{x^3 + C}$ is a solution to the differential equation $\dfrac{dy}{dx} = -x^2y^2$. Compare the graphs of this function for several values of C.

3. One model for the growth of a population is based on the assumption that **the population grows at a rate proportional to the size of the population**. This is a reasonable assumption for a population with unrestricted growth – unlimited environment, adequate nutrition, absence of predators, etc. – a population of bacteria, for example.

   (a) Let P denote the number of individuals in the population and let t denote the time. Using the fact that the derivative is a **rate of change**, write a differential equation to model this situation.

   (b) Verify that the family of functions $P = Ce^{kt}$ is a solution to the differential equation $\dfrac{dP}{dt} = kP$.

   (c) What is the significance of $P(0)$?

4. Unlimited growth for a population is not realistic; a more realistic model must reflect the fact that there are limited resources. Many populations start by increasing exponentially when the population is small but then level off when the population approaches its **carrying capacity** M. A good differential equation to model this situation is $\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)$, called the **logistic** differential equation.

   (a) What happens to this differential equation when P is small?

   (b) What happens to this differential equation when P is close to, but less than, M?

   (c) What happens to this differential equation when P is greater than M?

   (d) When is the population increasing?

   (e) When is the population decreasing?

   (f) What are the equilibrium solutions (when the population stays the same)?