

### 11.9: Representations of Functions as Power Series

In our last class, we considered power series, radii of convergence, and intervals of convergence. Now, we are going to represent **functions** using power series. For example, consider the series  $\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \dots$ . This is **geometric** series with  $a = 1$  and  $r = x$ . Thus, the series converges for  $|x| < 1$  so that the radius of convergence for this series is  $R = 1$ . It turns out that the interval of convergence is  $I = (-1, 1)$ . However, because this is a **geometric** series we can also find its sum which is a **function**:

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \dots = \frac{1}{1-x},$$

for  $x$  in the interval  $(-1, 1)$ .

We are now going to find power series representations of functions.

- (1) Consider the function  $f(x) = \frac{1}{1+x}$ .
  - (a) Find  $a$  and  $r$  so that  $f(x) = \frac{a}{1-r}$ .
  - (b) Using the values of  $a$  and  $r$  from the previous question, find the corresponding geometric series.
  - (c) Note that the series you found in the previous question is a **power series**. Determine the radius and interval of convergence for your power series.

(2) Consider the function  $f(x) = \frac{x}{9 + x^2}$ .

(a) Find  $a$  and  $r$  so that  $f(x) = \frac{a}{1 - r}$ .

(b) Using the values of  $a$  and  $r$  from the previous question, find the corresponding geometric series.

(c) Note that the series you found in the previous question is a **power series**. Determine the radius and interval of convergence for your power series.

We have used geometric series to power series representations of known functions. We can also **differentiate** and **integrate** our functions and series to find series representations of other functions.

(3) Consider the fact that  $\frac{1}{1 - x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \dots$ .

(a) Integrate both sides of this equation. For what function do we now have a power series representation? What are the radius and interval of convergence?

- (b) Differentiate both sides of this equation. For what function do we now have a power series representation? What are the radius and interval of convergence?

In the previous question, notice that the radius of convergence did not change when we integrated or differentiated. In fact, this is always true.

**Theorem 1** *If*

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \dots$$

*is a power series with radius of convergence  $R > 0$ , then  $f$  is continuous and differentiable on the interval  $(a-R, a+R)$  and*

(1)

$$\begin{aligned} f'(x) &= c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + 4c_4(x-a)^3 + \dots \\ &= \sum_{n=1}^{\infty} nc_n(x-a)^{n-1} \end{aligned}$$

*with radius of convergence  $R$ , and*

(2)

$$\begin{aligned} \int f(x) dx &= C + c_0(x-a) + c_1 \frac{(x-a)^2}{2} + c_2 \frac{(x-a)^3}{3} + c_3 \frac{(x-a)^4}{4} + \dots \\ &= C + \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1} \end{aligned}$$

*with radius of convergence  $R$ .*

