

## 11.2: Infinite Series and Geometric Series

**Definition 1** Consider

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \cdots + a_n + \cdots$$

a sum of an infinite number of terms, if such a sum even exists. Let

$$s_n = a_1 + a_2 + \cdots + a_n.$$

We call  $s_n$  the  $n$ th *partial sum* and the sequence  $\{s_n\}$  is called the *sequence of partial sums*. Then

$$\sum_{n=1}^{\infty} a_n = \lim_{n \rightarrow \infty} s_n.$$

We say that  $\sum_{n=1}^{\infty} a_n$  *converges* to  $s$  if  $\lim_{n \rightarrow \infty} s_n = s$  and  $\sum_{n=1}^{\infty} a_n$  *diverges* if  $\lim_{n \rightarrow \infty} s_n$  does not exist. So, the series  $\sum_{n=1}^{\infty} a_n$  and the sequence of partial sums  $\{s_n\}$  either both *converge* or both *diverge*.

For each of the series below, find the partial sums  $s_1, s_2, \dots, s_{10}$ . Based on your work, do you think the series converges or diverges? Explain.

(1)  $\sum_{n=1}^{\infty} \frac{1}{2^n}$

(2)  $\sum_{n=1}^{\infty} \frac{1}{n}$

(3)  $\sum_{n=1}^{\infty} \left( \frac{1}{n} - \frac{1}{n+1} \right)$

Some series are so special they are given special names.

**Definition 2** Consider

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \cdots + ar^n + \cdots$$

is a *geometric series* and this series converges to  $\frac{a}{1-r}$  if and only if  $|r| < 1$ .

We are going to prove the above fact about geometric series  $\sum_{n=1}^{\infty} ar^{n-1}$ , namely that a geometric series converges to  $a/(1-r)$  if  $|r| < 1$ .

- (1) Write out the  $n$  terms in the  $n$ th partial sum  $s_n$  for the geometric series  $\sum_{n=1}^{\infty} ar^{n-1}$ .
- (2) Write out the  $n$  terms in  $rs_n$ , that is, take your answer from the previous question and multiply by  $r$ .
- (3) Determine  $s_n - rs_n$  using your answers to the previous two questions.
- (4) Using your answer in (3), solve for  $s_n$ .
- (5) Now determine  $\lim_{n \rightarrow \infty} s_n$ . You should consider two cases: (1)  $|r| < 1$  and (2)  $|r| > 1$ .
- (6) Thus, when does the geometric series  $\sum_{n=1}^{\infty} ar^{n-1}$  converge? When it does converge, what's its sum?

Using your work on the previous page, determine whether the following geometric series are convergent or divergent. If a geometric series is convergent, find its sum.

(1)  $3 + 2 + \frac{4}{3} + \frac{8}{9} + \dots$

(2)  $\sum_{n=1}^{\infty} \frac{10^n}{(-9)^{n-1}}$

(3) Find the values of  $x$  for which the series  $\sum_{n=1}^{\infty} \frac{x^n}{3^n}$  converges.

(4) Show that  $0.\overline{9} = 0.99999999\dots = 1$  by writing  $0.\overline{9} = 0.99999999\dots$  as a geometric series and finding its sum.