Math 146
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Section 7.8: Improper Integrals, Type II

Definition 1 An *improper integral of type II* is a definite integral of the form $\int_a^b f(x)\,dx$ where f is discontinuous at $x = b$.
If $\int_a^t f(x)\,dx$ exists for every t with $a \le t < b$, then

$$\int_a^b f(x)\,dx = \lim_{t \to b^-} \int_a^t f(x)\,dx.$$

If this limit exists, we say the improper integral *converges;* otherwise the improper integral *diverges.*

Another example of an improper integral is $\int_a^b f(x)\,dx$ with f discontinuous at $x = a$ which we define as $\displaystyle\int_a^b f(x)\,dx = \lim_{t \to a^+} \int_t^b f(x)\,dx$. If f is discontinuous at some point c in $[a, b]$, to evaluate $\int_a^b f(x)\,dx$, we consider **two** improper integrals $\int_a^c f(x)\,dx$ and $\int_c^b f(x)\,dx$, and $\int_a^b f(x)\,dx$ converges to the sum of these two improper integrals, provided they both converge and diverges otherwise.

(1) Consider $\displaystyle\int_2^3 \frac{1}{\sqrt{3-x}}dx$.

(a) Is this an improper integral? Why or why not?

(b) Using the definition above, set up the appropriate limit that would need to be evaluated to determine whether this improper integral converges.

(c) Notice that in your answer to the previous question, you have both a limit and a definite integral to evaluate. Which must you determine first? Do so below.

(d) Now take the limit. Does this improper integral converge or diverge?

(2) Consider $\displaystyle\int_6^8 \frac{4}{(x-6)^3}\,dx$.

(a) Using the definition above, set up the appropriate limit that would need to be evaluated to determine whether this improper integral converges.

(b) Again, you have a definite integral and a limit to evaluate. Evaluate the definite integral below first.

(c) Now take the limit. Does this improper integral converge or diverge?

(3) Determine whether $\displaystyle\int_0^1 \frac{1}{4y-1}\,dy$ converges or diverges. If this improper integral converges, find it's value.

Comparison Theorem Suppose that f and g are functions such that $0 \leq f(x) \leq g(x)$ for all $x \geq a$.

- If $\int_a^\infty g(x)\,dx$ **converges**, then $\int_a^\infty f(x)\,dx$ **converges**.
- If $\int_a^\infty f(x)\,dx$ **diverges**, then $\int_a^\infty g(x)\,dx$ **diverges**.

(1) Consider $\int_1^\infty e^{-x^2}\,dx$.

(a) Based on all the examples you've seen so far, make a guess as to whether or not this integral converges or diverges.

(b) Based on your guess, do you need to find a function that is smaller than $\int_1^\infty e^{-x^2}\,dx$ or bigger?

(c) Find the appropriate such function.

(d) Use the Comparison Theorem to justify your guess as to whether or not the integral converges.

(2) Consider $\int_1^\infty \frac{1+e^{-x}}{x}dx$.

(a) Based on all the examples you've seen so far, make a guess as to whether or not this integral converges or diverges.

(b) Based on your guess, do you need to find a function that is smaller than $\int_1^\infty \frac{1+e^{-x}}{x}dx$ or bigger?

(c) Find the appropriate such function.

(d) Use the Comparison Theorem to justify your guess as to whether or not the integral converges.