11.3: The Integral Test

Consider the series $\sum_{n=1}^{\infty} \frac{1}{n}$ and the improper integral $\int_1^{\infty} \frac{1}{x}\, dx$. Recall that in class, by dividing the area under the curve $y = 1/x$ on the interval $[1, \infty)$ into segments $[1, 2], [2, 3], [3, 4], \ldots$ and drawing rectangles using the left-hand endpoint of each interval, we showed that $\int_1^{\infty} \frac{1}{x}\, dx < \sum_{n=1}^{\infty} \frac{1}{n}$ so that since $\int_1^{\infty} \frac{1}{x}\, dx$ diverges, then $\sum_{n=1}^{\infty} \frac{1}{n}$ also diverges. We are now going to consider the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ and show that it converges using the improper integral $\int_1^{\infty} \frac{1}{x^2}\, dx$.

(1) Draw the graph of $y = 1/x^2$ on the interval $[1, \infty)$.

(2) On the graph above, divide the interval $[1, \infty)$ into segments $[1, 2], [2, 3], [3, 4], [4, 5], \ldots$.

(3) Now draw rectangles using the **right-hand endpoint** of each interval. On the graph above, write in the area of the first rectangle, the second, the third, ..., the nth, ...

(4) Describe the area of these rectangles using an infinite sum (be careful about where this series starts). How does your infinite sum compare with $\int_1^{\infty} \frac{1}{x^2}\, dx$ (in value, i.e., which is larger)?

(5) So what can you conclude about the convergence or divergence of the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$?

We can use the same approach for any series $\sum_{n=1}^{\infty} a_n$, that is, we can compare the infinite series $\sum_{n=1}^{\infty} a_n$ with the improper integral $\int_1^{\infty} f(x)\, dx$ where $f(n) = a_n$ to decide whether or not a given series converges or diverges. This test is called the **Integral Test**.

Integral Test: Let $\sum_{n=1}^{\infty} a_n$ be a series with **positive** terms and let f be a function such that $f(n) = a_n$.

- If $\int_1^{\infty} f(x)dx$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.
- If $\int_1^{\infty} f(x)dx$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.

(1) When does the improper integral $\int_1^{\infty} \frac{1}{x^p}\, dx$ for a real number p converge, that is, for what values of p does this integral converge? For what values of p does this improper integral diverge?

(2) Consider the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ where p is a real number. Using your answer to the previous question, for what values of p does this series converge? For what values of p does this series diverge?

(3) Use the Integral Test to determine whether the series is convergent or divergent:

(a) $\sum_{n=1}^{\infty} ne^{-n}$

(b) $\sum_{n=1}^{\infty} \frac{n^2}{n^3+1}$

(c) $\sum_{n=1}^{\infty} \frac{1}{n^2+4}$