9.5: Linear Equations

In Section 9.5, we will learn another technique for algebraically solving differential equations. In this section we will use the notation y' for the derivative of the function $y = f(x)$ rather than $\frac{dy}{dx}$ as we did in Section 9.3.

The new technique will solve **first-order linear** differential equations. A **first-order** differential equation is one that involves the function y and its derivative y' only. For a first-order differential equation to be **linear**, it can be put into the form

$$y' + P(x)y = Q(x),$$

where P and Q are continuous functions of x. Notice that this differential equation is not separable because it's impossible to factor the expression for y' as a function of x times a function of y.

(1) Consider the differential equation $y' + 3x^2y = 6x^2$.

(a) Is this differential equation a first-order linear equation? If so, what is $P(x)$? $Q(x)$?

We will now solve this differential equation in the following steps.

(b) Find $\int P(x)\ dx$.

(c) Let $I(x) = e^{\int P(x)\ dx + C}$. If we let $C = 0$, what is $I(x)$?

(d) Now that you have found $I(x)$, multiply the differential equation by $I(x)$. We call $I(x)$ the **integrating factor**.

(e) Find $\int I(x)Q(x)\ dx$. Notice that you are integrating the right-hand side of the equation found in (d).

(f) Look at the left-hand side of the equation given in (d), namely the expression $I(x)y' + I(x)P(x)y$. Can you find an expression in x and y so that $I(x)y' + I(x)P(x)y$ is the derivative of your expression? (**Hint:** Think 'product rule'.)

(g) So, your answer to question (f) is equal to your answer to question (e) because to find each of these answers, you integrated both sides of the equation in (d). Using the fact that (e) = (f), find y as a function of x.

Thus the solution to the differential equation $y' + P(x)y = Q(x)$ is $\boxed{I(x)y = \int I(x)Q(x)\,dx}$ where $\boxed{I(x) = e^{\int P(x)\,dx}}$. To solve for y, we just need to divide both sides by $I(x)$ after integrating the right-hand side.

(2) Consider the initial-value problem $x^2y' + xy = 1$ with $x > 1$ and $y(1) = 2$.

(a) Put the differential equation into the form $y' + P(x)y = Q(x)$.

(b) Find $I(x)$, the integrating factor. Recall $I(x) = e^{\int P(x)\,dx}$. For this integration, let $C = 0$.

(c) Find $\int I(x)Q(x)\,dx$. Be careful here and don't forget the constant C!

(d) Using your work from the previous question, use the equation $I(x)y = \int I(x)Q(x)\,dx$ to find y as a function of x.

(e) Use the fact that $y(1) = 2$ to find the particular solution we need in this case.

(3) Consider the differential equation $y' + \cos x = y$.

(a) Put the differential equation into the form $y' + P(x)y = Q(x)$.

(b) Find $I(x)$, the integrating factor. Recall $I(x) = e^{\int P(x)\, dx}$. For this integration, let $C = 0$.

(c) Find $\int I(x)Q(x)\, dx$. Be careful here and don't forget the constant C!

(d) Using your work from the previous question, use the equation $I(x)y = \int I(x)Q(x)\, dx$ to find y as a function of x.