Math 146
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Section 7.4: Partial Fractions

Given $\int \frac{p(x)}{q(x)}$ where $p(x)$ and $q(x)$ are polynomials:

Step 1: Divide $q(x)$ into $p(x)$ if the degree of $p(x)$ is greater than or equal to the degree of $q(x)$.

Step 2: Factor q(x) into **linear factors** $(ax+b)^m$ and **quadratic factors** $(ax^2+bx+c)^n$.

Step 3: For each **linear factor** $(ax+b)^m$, include the following sum:

$$\frac{A_1}{(ax+b)} + \frac{A_2}{(ax+b)^2} + \frac{A_3}{(ax+b)^3} + \cdots + \frac{A_m}{(ax+b)^m}.$$

Step 4: For each **quadratic factor** $(ax^2+bx+c)^n$, include the following sum:

$$\frac{B_1x+C_1}{(ax^2+bx+c)} + \frac{B_2x+C_2}{(ax^2+bx+c)^2} + \frac{B_3x+C_3}{(ax^2+bx+c)^3} + \cdots + \frac{B_nx+C_n}{(ax^2+bx+c)^n}.$$

Step 5: Find the values of $A_1, A_2, \ldots, A_m, B_1, B_2, \ldots, B_n, C_1, C_2, \ldots, C_n$ from the expression

$$\begin{aligned}\frac{p(x)}{q(x)} &= \frac{A_1}{(ax+b)} + \frac{A_2}{(ax+b)^2} + \frac{A_3}{(ax+b)^3} + \cdots + \frac{A_m}{(ax+b)^m} + \cdots + \\ &\quad + \frac{B_1x+C_1}{(a'x^2+b'x+c')} + \frac{B_2x+C_2}{(a'x^2+b'x+c')^2} + \frac{B_3x+C_3}{(a'x^2+b'x+c')^3} + \cdots + \frac{B_nx+C_n}{(a'x^2+b'x+c')^n} + \cdots.\end{aligned}$$

Step 6: Integrate each fraction in the expression above to evaluate $\int \frac{p(x)}{q(x)}\,dx$.

Evaluate the following integrals by answering the following questions.

1. Consider $\int \frac{3x-1}{x^2-1}\,dx$.

   (a) Does Step 1 above apply here? Why or why not?

   (b) Factor the denominator x^2-1 into linear and/or quadratic factors.

   (c) Set up the appropriate partial fraction for each of the factors you found in the previous question, using Steps 3 and 4 above.

(d) Set your expression in the previous question equal to $\frac{3x-1}{x^2-1}$ to find the values of all constants as in Step 5 above.

(e) Finally, using your expression from the previous question, find $\int \frac{3x-1}{x^2-1}\,dx$.

2. Consider $\int \frac{2x^3+x^2-7x+7}{x^2+x-2}\,dx$.

(a) Now Step 1 above applies. Divide x^2+x-2 into $2x^3+x^2-7x+7$, i.e., find $r(x)$ and $p(x)$ so that $\frac{2x^3+x^2-7x+7}{x^2+x-2} = r(x) + \frac{p(x)}{x^2+x-2}$

(b) Factor the denominator x^2+x-2 into linear factors.

(c) Set up the appropriate partial fraction for each of the factors you found in the previous question, using Steps 3 and 4 above.

(d) Set your expression in the previous question equal to $\frac{p(x)}{x^2+x-2}$ to find the values of all constants as in Step 5 above.

(e) Finally find $\displaystyle\int \frac{2x^3 + x^2 - 7x + 7}{x^2 + x - 2}\, dx.$

3. Consider $\displaystyle\int \frac{6x - 11}{(x-1)^2}\, dx.$

(a) Set up the appropriate partial fractions, using Steps 3 and 4 above.

(b) Set your expression in the previous question equal to $\dfrac{6x-11}{(x-1)^2}$ to find the values of all constants as in Step 5 above.

(c) Finally, find $\displaystyle\int \frac{6x - 11}{(x-1)^2}\, dx$

4. Consider $\int \frac{4x^3 - 3x^2 + 6x - 27}{x^4 + 9x^2}\, dx$.

(a) Factor the denominator $x^4 + 9x^2$ into linear and/or quadratic factors.

(b) Set up the appropriate partial fraction for each of the factors you found in the previous question, using Steps 3 and 4 above.

(c) Set your expression in the previous question equal to $\frac{4x^3 - 2x^2 + 6x - 27}{x^4 + 9x^2}$ to find the values of all constants as in Step 5 above.

(d) Finally, using your expression from the previous question, find $\int \frac{4x^3 - 3x^2 + 6x - 27}{x^4 + 9x^2}\, dx$.