Math 146
H. Jordon

9.4: Two Models for Population Growth

In Section 9.4, we are going to investigate two models for population growth: **the law of natural growth** and **the logistic equation.**

Let P denote the size of a given population, for example, the number of bacteria in a petri dish, and let t denote time. Then $\frac{dP}{dt}$ is the rate of growth of the population with respect to time. If a population grows in proportional to its size, then we know $\frac{dP}{dt} = kP$ where k is a constant.

(1) Solve the differential equation $\frac{dP}{dt} = kP$ to find the function $P(t)$.

(2) Suppose you are told that the initial population is P_0, that is, $P(0) = P_0$. How does that affect the answer you gave in the previous question?

(3) Summarize the previous two questions by completing the following: The solution to the initial-value problem $\frac{dP}{dt} = kP$ with $P(0) = P_0$ is :

(4) A bacterial culture starts with 500 bacteria and doubles in size every half hour.

(a) Use your work above to find a function $P(t)$ giving the number of bacteria after t hours.

(b) How many are there after 40 minutes?

(c) When does the population reach 100,000?

The law of nature growth is not a realistic model because typically, a population will grow until it exhausts its resources, that is, at first, the population will grow exponentially, but, as it approaches its **carrying capacity**, it will level off due to limited resources. The **carrying capacity** M is the maximum population the given environment is capable of sustaining in the long run. In this case, population growth is modeled by the **logistic differential equation**

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right).$$

Notice that when P is small compared to M, then $1 - P/M \approx 1$ so that $\frac{dP}{dt} \approx kP$ while if P is close to M, then $1 - P/M \approx 0$ so that $\frac{dP}{dt} \approx 0$. It turns out that using separation of variables and partial fractions, we can solve this differential equation with initial value $P(0) = P_0$ to show that

$$P(t) = \frac{M}{1 + Ae^{-kt}} \text{ where } A = \frac{M - P_0}{P_0}.$$

Suppose that a certain population has an initial size of 200 and that after the first year, there are 1000. Assume that the carrying capacity of this population is 10,000 and that this population satisfies the logistic equation.

(1) Determine P(0) and P(1), where $P(t)$ is the size of the population and t is time in years.

(2) Determine the function $P(t)$.

(3) How long will it take for the population to increase to 5000?