Math 146
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Section 8.5: Probability

Probability measures the chances of an **event** happening. For example, suppose you roll a fair six-sided die (the **event** in this case). Let X denote the up-face on the die after it is rolled. We call X the **random variable**. Then, the possible **outcomes** or values of X are $1, 2, 3, 4, 5, 6$. Since the die is fair, you have a $1/6$ chance of getting each side. Thus, we say

$$P(X = 1) = P(X = 2) = P(X = 3) = P(X = 4) = P(X = 5) = P(X = 6) = 1/6.$$

Some things to notice here:
(1) $P(X = 1) + P(X = 2) + P(X = 3) + P(X = 4) + P(X = 5) + P(X = 6) = 1$, and
(2) X is a **discrete** random variable because the possible values for X are **finite**. In Section 8.5, we are going to study **continuous** random variables.

Suppose our event is to measure the life of a car battery. Let X be the amount of time, in hours, that the battery lasts. Then X is not a discrete random variable because any nonnegative value is possible for X. Also, the probability that a battery will last a certain length of time is not so easily defined. We need something called a **probability density function** $f(x)$ for X, where the area under the curve of $y = f(x)$ on the interval $[a, b]$ will be $P(a \leq X \leq b)$. So, when dealing with a continuous random variable, we need both the variable X and its probability density function $f(x)$, and the function $f(x)$ must satisfy the following definition.

Definition 1 Let X be a continuous random variable and let $f(x)$ be the probability density function for X. Then

(1) $\displaystyle\int_{-\infty}^{\infty} f(x)\ dx = 1$

(2) $\displaystyle P(a \leq X \leq b) = \int_a^b f(x)\ dx$

Answer each of the following:

(1) Let X be the length of time, in hours, required by students to complete a 1-hour exam. Then X is a continuous random variable. Suppose X has probability density function

$$f(x) = \begin{cases} cx^2 + x & 0 \leq x \leq 1 \\ 0 & x > 1 \text{ or } x < 0. \end{cases}$$

(a) Find the value of c so that $f(x)$ is a probability density function. (**Hint:** See (1) in the definition above.)

(b) Using the value of c you found above, graph the function $f(x)$ with the appropriate window.

(c) Suppose we wish to find the probability that a student finishes in 30 minutes or less. First, draw a graph to represent this situation. Second, set up and evaluate a definite integral giving this probability.

(d) What is the probability that a student finishes in more than 30 minutes?

Some continuous random variables occur so often they are given special names. We are going to study two of these continuous random variables, the **exponential random** variable and the **normal** random variable.

The **exponential** random variable is useful for modeling the length of time before a certain event happens, like the life of a battery or waiting in line, etc. To describe an exponential random variable, we are given the **mean** or average length of time for X; for example, if X is the length of time a battery lasts, we would also need to know that on average, this particular car battery lasts 180 hours. We say that 180 is the mean, or in this case, that $\mu = 180$ in the definition below.

Definition 2 Let X be an *exponential random variable* with mean μ. Then the *exponential probability density function* for X is $f(x) = \begin{cases} \frac{1}{\mu}e^{-x/\mu} & x \geq 0 \\ 0 & x < 0. \end{cases}$

Answer the following questions.

(1) The magnitudes of earthquakes recorded in a region of North America can be modeled as having an exponential distribution with mean $\mu = 2.4$, as measured on the Richter scale.

(a) Find and graph the probability density function for the random variable X, the magnitude of an earthquake in this region.

(b) Draw a graph to represent the probability that an earthquake in this region will exceed 3.0 on the Richter scale. Determine this probability be setting up and evaluating a definite integral.

(c) Draw a graph to represent the probability that an earthquake striking this region will fall between 2.0 and 3.0 on the Richter scale. Determine this probability be setting up and evaluating a definite integral.

The **normal** random variable has a probability density function that is shaped like a bell-curve. Examples of normal random variable are test scores on aptitude tests, heights and weights of individuals from a homogeneous population, annual rainfall in a given location. For the normal random variable X, we must be given the **mean** μ and the **standard deviation** σ, which measures how spread out the values of X are.

Definition 3 Let X be a *normal random variable* with mean μ and standard deviation σ. Then the *normal probability density function* for X is

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/(2\sigma^2)}.$$

(2) The grade point averages of a large population of college students are approximately normally distributed with a mean $\mu = 2.4$ and standard deviation $\sigma = 0.8$.

(a) Find and graph the probability density function for the random variable X, the grade point average.

(b) Draw a graph to represent the probability that that a student will have a grade point average greater than 3.0. Determine this probability be setting up and evaluating a definite integral.

(c) Suppose the university has the policy that students whose grade point averages are less than 1.9 are dropped from the university. Draw a graph to represent the percentage of students who are in danger of being dropped. Determine this probability be setting up and evaluating a definite integral.