11.6: Absolute Convergence and the Ratio Test

Recall that in class on Wednesday, we consider series with both **positive** and **negative** terms.

(1) Give an example of a series $\sum_{n=1}^{\infty} a_n$ such that $\sum_{n=1}^{\infty} a_n$ converges yet $\sum_{n=1}^{\infty} |a_n|$ diverges.

(2) Give an example of an alternating series $\sum_{n=1}^{\infty} a_n$ such that both series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} |a_n|$ converge.

Thus, sometimes a series is dependent on having negative terms to converge and sometimes not. When a series does not need those terms to be negative, we say that it **absolutely**.

Definition 1 Let $\sum_{n=1}^{\infty} a_n$ be a series which consists of positive and possibly negative terms. Then $\sum_{n=1}^{\infty} a_n$ is *absolutely convergent* if $\sum_{n=1}^{\infty} |a_n|$ converges.

So the series you gave in Question 2 above is **absolutely convergent** while the series you gave in Question 1 is not. The series in Question 1 is **conditionally convergent**.

Definition 2 A series $\sum_{n=1}^{\infty} a_n$ is *conditionally convergent* if $\sum_{n=1}^{\infty} a_n$ converges and $\sum_{n=1}^{\infty} |a_n|$ diverges.

Note that the definition of absolute convergence says nothing about whether or not the series itself actually converges. However, it shouldn't be too surprising that if a series is absolutely convergent, it is also convergent.

Theorem 3 *If* $\sum_{n=1}^{\infty} a_n$ *is absolutely convergent, then* $\sum_{n=1}^{\infty} a_n$ *is convergent.*

So, absolute convergence is **stronger** than convergence; that is, to say that a series is absolutely convergent is stronger than saying it converges.

The Ratio Test can be used to determine if a series is absolutely convergent.

Ratio Test: Let $\sum_{n=1}^{\infty} a_n$ be a series such that $\lim_{n\to\infty} a_n = 0$. If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L$, then

- $L < 1$ gives absolute convergence,
- $L > 1$ gives divergence, and
- $L = 1$ gives NO information.

Basically, what the Ratio Test says about the series $\sum_{n=1}^{\infty} |a_n|$ is that it behaves "like" the geometric series $\sum_{n=1}^{\infty} L^n$, which we know converges if $L < 1$ and diverges if $L > 1$. When $L = 1$, the Ratio Test **fails** to provide any information about the series and we need to **use another test**, such as the Integral Test or Comparison Test, etc.

For each of the series below, determine whether the series is absolutely convergent, conditionally convergent, or divergent by answering the following questions.

(1) $\sum_{n=1}^{\infty} \frac{(-10)^n}{n!}$

(a) What's a_n? a_{n+1}?

(b) Determine $\left|\frac{a_{n+1}}{a_n}\right|$. Simplify.

(c) Find $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$. What conclusion, if any, can you draw from the Ratio Test?

(2) $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{100^n}$

(a) What's a_n? a_{n+1}?

(b) Determine $\left|\frac{a_{n+1}}{a_n}\right|$. Simplify.

(c) Find $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$. What conclusion, if any, can you draw from the Ratio Test?

(3) $\displaystyle\sum_{n=1}^{\infty} \frac{10^n}{(n+1)4^{2n+1}}$

(a) What's a_n? a_{n+1}?

(b) Determine $\left|\frac{a_{n+1}}{a_n}\right|$. Simplify.

(c) Find $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$. What conclusion, if any, can you draw from the Ratio Test?

(4) $\sum_{n=1}^{\infty}(-1)^n\frac{n}{\sqrt{n^3+2}}$

(a) What's a_n? a_{n+1}?

(b) Determine $\left|\frac{a_{n+1}}{a_n}\right|$. Simplify.

(c) Find $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$. What conclusion, if any, can you draw from the Ratio Test?

(d) Since the Ratio Test fails in this case, we need to use one of the other tests: Integral, Comparison, etc., to decide if the series is absolutely convergent. Using one of those tests, determine if $\sum_{n=1}^{\infty}\left|(-1)^n\frac{n}{\sqrt{n^3+2}}\right| = \sum_{n=1}^{\infty}\frac{n}{\sqrt{n^3+2}}$ converges. What can you say about the absolute convergence of $\sum_{n=1}^{\infty}(-1)^n\frac{n}{\sqrt{n^3+2}}$? Is $\sum_{n=1}^{\infty}(-1)^n\frac{n}{\sqrt{n^3+2}}$ conditionally convergent?

(5) $\sum_{n=1}^{\infty} (-1)^n \frac{2^n n!}{5 \cdot 8 \cdot 11 \cdots (3n+2)}$

(a) What's a_n? a_{n+1}?

(b) Determine $\left|\frac{a_{n+1}}{a_n}\right|$. Simplify.

(c) Find $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$. What conclusion, if any, can you draw from the Ratio Test?