Math 146
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9.3: Separation of variables

In Section 9.3, we are going to learn how to solve differential equations using a technique called **separation of variables**. It's important to note that this technique will not **always** work, but does work in a significant number of cases.

Suppose we are given a first-order differential equation

$$\frac{dy}{dx} = h(x, y).$$

If we can re-write $h(x, y) = f(x)g(y)$, so that

$$\frac{dy}{dx} = f(x)g(y),$$

then we **separate** the variables to obtain

$$\frac{1}{g(y)}\ dy = f(x)\ dx.$$

Next, we integrate both sides

$$\int \frac{1}{g(y)}\ dy = \int f(x)\ dx + C$$

and then solve for y, if possible. Notice that we put the constant C on the right.

(1) Consider the differential equation $xy' + y = y^2$.

(a) **Separate** the equation to write the equations as $h(y)\ dy = f(x)\ dx$ for some $h(y)$ and $f(x)$. Start by solving for y' and then replacing y' with $\frac{dy}{dx}$.

(b) **Integrate** both sides of your equation in the previous question. Don't forget "$+C$" on the right!

(c) **Solve** your equation in the previous question for y.

(d) Find the **specific solution** that passes through the point $(1, -1)$.

(e) Verify that the specific solution you found in the previous question satisfies the given differential equation.

(2) Suppose you have just poured a cup of freshly brewed coffee with temperature 95°C in a room where the temperature is 20°C. Newton's Law of Cooling state that the rate of cooling of an object is proportional to the temperature difference between the object and its surroundings, provided that this difference is not too large.

(a) Let y denote the temperature of the coffee, in °C, and let t denote the time in minutes. In terms of t and y, what does it mean to say that the coffee was 95° when it was first poured?

(b) Use Newton's Law of Cooling to find a differential equation for this particular situation.

(c) Solve your differential equation.

(d) What is the temperature of the coffee after 30 minutes (assuming the cup was poured and then went undisturbed)? In the long run?

(3) A tank contains 20 kg of salt dissolved in 5000 L of water. Brine that contains 0.03 kg of salt per liter of water enters the tank at a rate of 25 L/min. The solution is kept thoroughly mixed and drains from the tank at the same rate.

(a) Let y denote the amount of salt, in kg, in the tank and let t denote the time in minutes. What does it mean that 20 kg of salt was dissolved in the tank initially?

(b) Using the fact that the rate of change in the amount of salt in the tank with respect to time is **rate in - rate out**, find a differential equation $\frac{dy}{dt}$ to model this situation.

(c) Solve your differential equation.

(d) How much salt remains in the tank after a half an hour? In the long run?