Math 146
H. Jordon

11.10: Taylor Series

Recall the following:

Theorem 1 *If*

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$$

for $|x-a| < R$, *then*

$$c_n = \frac{f^{(n)}(a)}{n!}.$$

That is to say, if f has a power series representation at a, then it must be the following:

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \\ &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots \end{aligned}$$

The right hand side of the equation above is called **the Taylor series of the function** f **at** a. Most of the time, we choose $a = 0$ and thus we have the following:

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n \\ &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots \end{aligned}$$

Once we find Taylor series, we can do lots of amazing things with them, like evaluate limits, integrals, find Taylor series for other functions, etc. In what follows, you are going to use **known** Taylor series to do answer a variety of questions. Thus, it is important to remember a few Taylor series.

Here are some important series, together with their intervals of convergence, to remember:

Some Important Series:

- $\dfrac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n$ for $-1 < x < 1$
- $\ln(1-x) = -x - \dfrac{x^2}{2} - \dfrac{x^3}{3} - \cdots = -\sum_{n=1}^{\infty} \dfrac{x^n}{n}$ for $-1 \le x < 1$
- $\tan^{-1} x = x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \dfrac{x^7}{7} + \cdots = \sum_{n=0}^{\infty} (-1)^n \dfrac{x^{2n+1}}{2n+1}$ for $-1 \le x \le 1$
- $e^x = 1 + x + \dfrac{x^2}{2!} + \cdots = \sum_{n=0}^{\infty} \dfrac{x^n}{n!}$ for all x
- $\cos x = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \dfrac{x^6}{6!} \cdots = \sum_{n=0}^{\infty} (-1)^n \dfrac{x^{2n}}{(2n)!}$ for all x
- $\sin x = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dfrac{x^7}{7!} \cdots = \sum_{n=0}^{\infty} (-1)^n \dfrac{x^{2n+1}}{(2n+1)!}$ for all x

Answer each of the following questions:

(1) Use a **known** Taylor series to find a Taylor series and its radius of convergence for the given function.

(a) $f(x) = x^2 \tan^{-1}(x^3)$.

(b) $f(x) = \ln(1 + x^2)$

(2) Use a series to approximate the definite integral $\int_0^{0.5} x^2 e^{-x^2}\, dx$, accurate to three decimal places.

(3) Use a series to evaluate $\lim_{x \to 0} \dfrac{x - \tan^{-1} x}{x^3}$.

(4) Find the sum of the following series.

(a) $\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{x^{4n}}{n!}$

(b) $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n\pi^{2n+1}}{4^{2n+1}(2n+1)!}$

(c) $1-\ln 2+\dfrac{(\ln 2)^2}{2!}-\dfrac{(\ln 2)^3}{3!}+\cdots$