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Math 175

Invertible Matrix Theorem

Recall the following theorems from Sections 1.6 and 1.7:

Theorem 1.6, Span Theorem: Let A be an $m \times n$ matrix. TFAE:

1. the span of the columns of A is ____________
2. the equation $A\mathbf{x} = \mathbf{b}$ has __________________ for every $\mathbf{b} \in \mathbb{R}^m$
3. rank $A =$ ____________
4. there is a pivot position in each ____________ of A
5. rref(A) has no ____________ rows

Theorem 1.8, Linear Independence Theorem: Let A be an $m \times n$ matrix. TFAE:

1. the columns of A are __________________
2. the equation $A\mathbf{x} = \mathbf{b}$ has __________________ for every $\mathbf{b} \in \mathbb{R}^m$
3. rank $A =$ ____________
4. nullity $A =$ ____________
5. the only solution to $A\mathbf{x} = \mathbf{0}$ is __________________
6. there is a pivot position in each ____________ of A
7. columns of rref(A) are ________________________ in $\mathbb{R}^m$

Now, let's put this all together. Let's suppose A is an $n \times n$ invertible matrix. We've seen that this implies rref(A) $= I_n$.

1. What facts about A are true from Theorem 1.6?

2. What facts about A are true from Theorem 1.8?

Combining what you've written above together with a few other facts, we have:

Theorem 2.6, Invertible Matrix Theorem: Let A be an $n \times n$ matrix. TFAE:

1. A is ____________
2. $\text{rref}(A) = I_n$
3. the span of the columns of A is ____________
4. the equation $A\mathbf{x} = \mathbf{b}$ has __________________ for every $\mathbf{b} \in \mathbb{R}^m$
5. rank $A =$ ____________
6. the columns of A are __________________
7. nullity $A =$ ____________
8. the only solution to $A\mathbf{x} = \mathbf{0}$ is __________________
9. there exists an $n \times n$ matrix B such that $BA = I_n$
10. there exists an $n \times n$ matrix C such that $AC = I_n$
11. A is the product of elementary matrices

Items (1)–(8) all follow from Theorems 1.6 & 1.8 together with the fact that A is invertible iff $\text{rref}(A) = I_n$. It remains to show (9)–(11) are true.