

Section 7.2 Linear Transformations

Definition: Let V and W be vector spaces. A function $T : V \rightarrow W$ is a **linear transformation** if for all “vectors” u and v of V and scalars c , the following conditions hold:

$$(i) \quad T(u \oplus_V v) = T(u) \oplus_W T(v)$$

$$(ii) \quad T(c \odot_V u) = c \odot_W T(u)$$

The vector space V is called the **domain** and W is called the **codomain**.

Let $T : \mathcal{P}_2 \rightarrow \mathbb{R}^3$ be defined by $T(f(x)) = \begin{bmatrix} f(1) \\ f'(1) \\ f''(1) \end{bmatrix}$.

1. Suppose $f(x) = a_0 + a_1x + a_2x^2$. Find an expression for $T(f(x))$ in terms of the coefficients a_0, a_1, a_2 .

2. Show that T is a linear transformation. (**Hint:** Let $f(x) = a_0 + a_1x + a_2x^2$ and $g(x) = b_0 + b_1x + b_2x^2$. Show $T(f(x) \oplus_{\mathcal{P}_2} g(x)) = T(f(x)) + T(g(x))$ and $T(c \odot_{\mathcal{P}_2} f(x)) = cT(f(x))$.)

All the properties of linear transformations that we learned about in Chapter 2 are true now as well.

Theorem 7.4 Let V and W be vector spaces and let $T : V \rightarrow W$ be a linear transformation. Then for all “vectors” $u, v \in V$ and scalars $a, b \in \mathbb{R}$, the following statements are true:

- (a) $T(\mathbf{0}_V) = \mathbf{0}_W$
- (b) $T(-u) = -T(u)$
- (c) $T(u - v) = T(u) - T(v)$
- (d) $T(au + bv) = aT(u) + bT(v)$

Proof.

- (a) We want to show $T(\mathbf{0}_V) = \mathbf{0}_W$.

$$\begin{aligned} \mathbf{0}_V + \mathbf{0}_V &= \text{___} \text{ by Axiom ___ of a vector space} \\ T(\mathbf{0}_V) + T(\mathbf{0}_V) &= T(\text{___}) \\ T(\mathbf{0}_V) + T(\mathbf{0}_V) &= T(\text{___}) + \mathbf{0}_W \text{ by Axiom ___ of a vector space} \\ T(\mathbf{0}_V) &= \mathbf{0}_W \text{ by Theorem 7.2 ___ .} \end{aligned}$$

- (b) Now we show $T(-u) = -T(u)$. So, we have to show the equation $T(u) + \text{___} = \mathbf{0}_W$ holds. Then, by Theorem 7.2 ___, since inverses are unique, we conclude $-T(u) = \text{___}$.

$$\begin{aligned} T(u) + T(-u) &= T(\text{___} + \text{___}) \text{ since } T \text{ is a } \text{_____} \\ &= T(\text{___}) \text{ by Axiom ___ of a vector space} \\ &= \mathbf{0}_W \text{ by part ___} \end{aligned}$$

So $-T(u) = T(-u)$ by Theorem 7.2 ___.

Parts (c) and (d) are straightforward to verify. I encourage you to do so on your own.

Definition: Let V and W be vector spaces and $T : V \rightarrow W$ be a linear transformation. The **null space** of T is the set $\{v \in V \mid T(v) = \mathbf{0}_W\}$. The **range** of T is the set $\{T(v) \mid v \in V\}$.

It is not difficult to prove (and the proof is nearly the same as it was in Chapter 2) that the null space is a subspace of V and the range is a subspace of W .

1. For the linear transformation T defined above, find the null space of T .

2. Describe the range of T as the span of a set of vectors.

Definition: A linear transformation $T : V \rightarrow W$ is **one-to-one** if $v \neq w$ in V , then $T(v) \neq T(w)$ in W (if $T(\mathbf{v}) = T(\mathbf{w})$, then $\mathbf{v} = \mathbf{w}$) and is **onto** if the range of T is all of W , i.e, if the range is equal to the codomain.

It is straightforward to prove the following theorem:

Theorem 7.5 A linear-transformation $T : V \rightarrow W$ is one-to-one if and only if the null space of T is $\{\mathbf{0}_V\}$.

Proof. Let $T : V \rightarrow W$ be a linear transformation where V and W are vector spaces.

(\Rightarrow) Suppose T is one-to-one. This means that if $T(u) = T(v)$, then $u = v$. We want to show if $T(w) = \mathbf{0}_W$, then $w = \mathbf{0}_V$. This would imply that the null space of T is $\{\mathbf{0}_V\}$. So, suppose $T(w) = \mathbf{0}_W$ for some $w \in V$. But, $T(\mathbf{0}_V) = \mathbf{0}_W$ by Theorem 7.4 _____. So, $T(\mathbf{0}_V) = T(w)$ and since T is one-to-one, it must be that $w = \mathbf{0}_V$.

(\Leftarrow) Now suppose the null space of T is $\{\mathbf{0}_V\}$. We wish to show that T is one-to-one. So, we will show that if $T(u) = T(v)$, then $u = v$. So, suppose

$$\begin{aligned}
 T(u) &= T(v) \\
 T(\text{_____}) &= \mathbf{0}_W \text{ since } T \text{ is a } \text{_____} \\
 \text{_____} &= \mathbf{0}_V \text{ since the null space of } T \text{ is } \text{_____} \\
 \text{_____} &= \text{_____}
 \end{aligned}$$

Thus T is one-to-one.

Answer each of the following questions.

1. Do you think the linear transformation T defined above is one-to-one? Why or why not?

2. Do you think T is onto? Why or why not?

3. Let $T : \mathcal{M}_{2 \times 2} \rightarrow \mathcal{M}_{2 \times 2}$ be defined by $T(A) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} A$.

(a) Determine the null space of T . Is T one-to-one? Explain. (**Hint:** You may find it helpful to let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Solve $T(A) = \mathbf{0}_{\mathcal{M}_{2 \times 2}}$ by setting up a system of equations.)

(b) Determine the range of T . Is T onto? Explain. (**Hint:** Your work in the previous question will be helpful. Now the question becomes does $T(A) = B$ have a solution for every B in $\mathcal{M}_{2 \times 2}$?)