H. Jordon
Math 175

Matrix-Vector Products

Answer the following questions in groups of 3 to 4 people. Please use your textbook!

Definition: Let $A = [\mathbf{a}_1\mathbf{a}_2 \ldots \mathbf{a}_n]$ be a $m \times n$ matrix, where $\mathbf{a}_i$ denotes the ith column of A, and let $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ be a $n \times 1$ vector where each v_i is a scalar. The **matrix-vector product** $A\mathbf{v}$ is the $m \times 1$ vector found as follows:

$$A\mathbf{v} = v_1\mathbf{a}_1 + v_2\mathbf{a}_2 + \ldots v_n\mathbf{a}_n.$$

1. Give an example of a matrix-vector product with a 4×3 matrix A.

2. True or False: If A is a $m \times n$ matrix and $\mathbf{v}$ is a $n \times 1$ vectors, then $A\mathbf{v}$ is a linear combination of the columns of A. Explain.

3. Using the matrix A you gave in Question 1, determine $A\mathbf{e}_1, A\mathbf{e}_2, A\mathbf{e}_3$. In general, if A is an $m \times n$ matrix, what is $A\mathbf{e}_i$?

4. What is does the notation I_n mean? Find I_2, I_3, and I_4. Give an example of a 3×1 vector $\mathbf{v}$ and determine $I_3\mathbf{v}$.

5. Let $A = [\mathbf{a}_1\mathbf{a}_2 \ldots \mathbf{a}_n]$ and $B = [\mathbf{b}_1\mathbf{b}_2 \ldots \mathbf{b}_n]$ be $m \times n$ matrices and let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$. Complete the following:

   (a) $A(\mathbf{u} + \mathbf{v}) =$ ____________

   (b) $A(c\mathbf{u}) =$ __________ $=$ __________ for every $c \in \mathbb{R}$

   (c) $(A + B)\mathbf{u} =$ ____________

   (d) $A\mathbf{e}_j =$ ____________

   (e) If $A\mathbf{w} = B\mathbf{w}$ for every $\mathbf{w} \in \mathbb{R}^n$, then ____________.

   (f) $A\mathbf{0} =$ __________ where $\mathbf{0}$ is the $n \times 1$ vector with every coordinate 0

   (g) $\mathcal{O}\mathbf{v} =$ __________

   (h) $I_n\mathbf{v} =$ __________

6. Let's prove part (a) above by completing the following:

Let $A = [\underline{\hspace{6em}}]$ where $\underline{\hspace{2.5em}}$ denotes the ith column of A. Let $\mathbf{u} = \begin{bmatrix} \underline{\hspace{1.5em}} \\ \underline{\hspace{1.5em}} \\ \vdots \\ \underline{\hspace{1.5em}} \end{bmatrix}$

and $\mathbf{v} = \begin{bmatrix} \underline{\hspace{1.5em}} \\ \underline{\hspace{1.5em}} \\ \vdots \\ \underline{\hspace{1.5em}} \end{bmatrix}$. Then,

$$\begin{aligned}
A(\mathbf{u}+\mathbf{v}) &= A\left(\begin{bmatrix} \underline{\hspace{1.5em}} \\ \underline{\hspace{1.5em}} \\ \vdots \\ \underline{\hspace{1.5em}} \end{bmatrix} + \begin{bmatrix} \underline{\hspace{1.5em}} \\ \underline{\hspace{1.5em}} \\ \vdots \\ \underline{\hspace{1.5em}} \end{bmatrix}\right) \\
&= A\begin{bmatrix} \underline{\hspace{3em}} \\ \underline{\hspace{3em}} \\ \vdots \\ \underline{\hspace{3em}} \end{bmatrix} \quad \text{by the definition of } \underline{\hspace{6em}} \\
&= (\underline{\hspace{3em}})\mathbf{a}_1 + (\underline{\hspace{3em}})\mathbf{a}_2 + \cdots + (\underline{\hspace{3em}})\mathbf{a}_n \quad \text{by the definition of } \underline{\hspace{6em}} \\
&= (\underline{\hspace{1.5em}}\mathbf{a}_1 + \underline{\hspace{1.5em}}\mathbf{a}_2 + \cdots + \underline{\hspace{1.5em}}\mathbf{a}_n) + (\underline{\hspace{1.5em}}\mathbf{a}_1 + \underline{\hspace{1.5em}}\mathbf{a}_2 + \cdots + \underline{\hspace{1.5em}}\mathbf{a}_n) \quad \text{by Theorem 1.1 } \underline{\hspace{1.5em}} \\
&= A\underline{\hspace{1.5em}} + A\underline{\hspace{1.5em}} \quad \text{by the definition of } \underline{\hspace{6em}}
\end{aligned}$$

7. Now let's take a look at part (b). How many multiplications are done in the left-hand side of the equation? The center? The right-hand side of the equation?

Let's prove part (b) by completing the following:

Let $A = [\underline{\hspace{3cm}}]$ where $\underline{\hspace{1.2cm}}$ denotes the ith column of A. Let $\mathbf{u} = \begin{bmatrix} \underline{\hspace{0.6cm}} \\ \underline{\hspace{0.6cm}} \\ \vdots \\ \underline{\hspace{0.6cm}} \end{bmatrix}$.

Let $\underline{\hspace{0.6cm}}$ be a scalar.

$$
\begin{aligned}
A(c\mathbf{u}) &= A\left(c\begin{bmatrix} \underline{\hspace{0.6cm}} \\ \underline{\hspace{0.6cm}} \\ \vdots \\ \underline{\hspace{0.6cm}} \end{bmatrix}\right) \\
&= A\begin{bmatrix} \underline{\hspace{1.2cm}} \\ \underline{\hspace{1.2cm}} \\ \vdots \\ \underline{\hspace{1.2cm}} \end{bmatrix} \text{ by the definition of } \underline{\hspace{2.5cm}} \\
&= (\underline{\hspace{1.2cm}})\mathbf{a}_1 + (\underline{\hspace{1.2cm}})\mathbf{a}_2 + \cdots + (\underline{\hspace{1.2cm}})\mathbf{a}_n \text{ by the definition of } \underline{\hspace{2.5cm}} \\
&= \underline{\hspace{0.6cm}}(\underline{\hspace{0.6cm}}\mathbf{a}_1 + \underline{\hspace{0.6cm}}\mathbf{a}_2 + \cdots + \underline{\hspace{0.6cm}}\mathbf{a}_n) \text{ by Theorem 1.1 } \underline{\hspace{0.6cm}} \\
&= c(A\underline{\hspace{0.6cm}}) \text{ by the definition of } \underline{\hspace{2.5cm}}
\end{aligned}
$$

Proving that $A(c\mathbf{u}) = (cA)\mathbf{u}$ is similar. You should try it!