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Math 175

Matrix Multiplication

Answer the following questions in groups of 3 to 4 people. Please use your textbook!

Definition: Let $A = [\mathbf{a}_1\mathbf{a}_2 \ldots \mathbf{a}_n]$ be a $m \times n$ matrix, where $\mathbf{a}_i$ denotes the ith column of A, and let $B = [\mathbf{b}_1\mathbf{b}_2 \ldots \mathbf{b}_p]$ be a $n \times p$ matrix, where $\mathbf{b}_j$ denotes the jth column of B. The (matrix) **product** AB is the $m \times p$ matrix

$$AB = [A\mathbf{b}_1\ A\mathbf{b}_2 \ldots A\mathbf{b}_p].$$

Properties of Matrix Multiplication: Let A and B be $k \times m$ matrices, C be an $m \times n$ matrix, and P and Q are $n \times p$ matrices. Fill in the blanks below:

1. $s(AC) =$ ____________ $=$ ____________ for every $s \in \mathbb{R}$
2. $A(CP) =$ ____________
3. $(A + B)C =$ ____________
4. $C(P + Q) =$ ____________
5. $I_kA =$ ____________ $= AI_m$
6. $A\mathcal{O} =$ ____________
7. $(AC)^T =$ ____________

Let's prove part (3) above by completing the following:
Let $C = [$ _______________ $]$ where ______ denotes the ith column of C. Then,

$(A + B)C$ = $(A + B)[$ _______________ $]$
= $[(A + B)$___ $(A + B)$___ $\ldots (A + B)$___$]$ by the definition of ____________
= $[($___ $+$ ___$)$ $($___ $+$ ___$)$ $\cdots ($___ $+$ ___$)]$ by Theorem 1.3 ___
= $[$ ______________________ $] + [$ ______________________ $]$ by the definition of ____________
= A___ $+ B$___ by the definition of ____________

Let's prove part (4) above by completing the following:
Let $P = [\underline{\hspace{6em}}]$ where $\underline{\hspace{3em}}$ denotes the ith column of P and $Q = [\underline{\hspace{6em}}]$ where $\underline{\hspace{3em}}$ denotes the ith column of Q. Then,

$$\begin{aligned}
C(P+Q) &= C([\underline{\hspace{6em}}] + [\underline{\hspace{6em}}]) \\
&= C\,[(\underline{\hspace{3em}})\ (\underline{\hspace{3em}})\ \ldots (\underline{\hspace{3em}})] \text{ by the definition of } \underline{\hspace{6em}} \\
&= [C(\underline{\hspace{3em}})\ C(\underline{\hspace{3em}})\ \ldots C(\underline{\hspace{3em}})] \text{ by the definition of } \underline{\hspace{6em}} \\
&= [(\underline{\hspace{3em}} + \underline{\hspace{3em}})\ (\underline{\hspace{3em}} + \underline{\hspace{3em}})\ \cdots (\underline{\hspace{3em}} + \underline{\hspace{3em}})] \text{ by Theorem 1.3 } \underline{\hspace{2em}} \\
&= [\underline{\hspace{9em}}] + [\underline{\hspace{9em}}] \text{ by the definition of } \underline{\hspace{6em}} \\
&= C\underline{\hspace{2em}} + C\underline{\hspace{2em}} \text{ by the definition of } \underline{\hspace{6em}}
\end{aligned}$$

Let's prove part (1) by completing the following:
Let s be a scalar and let $C = [\underline{\hspace{6em}}]$ where $\underline{\hspace{3em}}$ denotes the ith column of C.

$$\begin{aligned}
s(AC) &= s(A[\underline{\hspace{9em}}]) \\
&= s([A\underline{\hspace{3em}}\ \ A\underline{\hspace{3em}}\ \ \cdots\ \ A\underline{\hspace{3em}}]) \text{ by the definition of } \underline{\hspace{6em}} \\
&= [s(A\underline{\hspace{3em}})\ \ s(A\underline{\hspace{3em}})\ \ \cdots s(A\underline{\hspace{3em}})] \text{ by the definition of } \underline{\hspace{6em}} \\
&= [(\underline{\hspace{3em}})\mathbf{c}_1\ \ (\underline{\hspace{3em}})\mathbf{c}_2\ \ \cdots (\underline{\hspace{3em}})\mathbf{c}_p] \text{ by Theorem 1.3 } \underline{\hspace{2em}} \\
&= (\underline{\hspace{3em}})[\underline{\hspace{9em}}] \text{ by the definition of } \underline{\hspace{6em}} \\
&= (\underline{\hspace{3em}})\underline{\hspace{2em}} \text{ by the definition of } \underline{\hspace{6em}}
\end{aligned}$$

Proving that $s(AC) = A(sC)$ is similar. You should try it!