Math 175
H. Jordon

Section 4.2: Basis and Dimension

Definition: Let V be a nonzero subspace of $\mathbb{R}^n$. A **basis** for V is a linearly independent generating set for V.

For example, $\mathcal{E} = \{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ is a basis for $\mathbb{R}^n$, called the **standard basis**.

Theorem

- Every nonzero subspace V of $\mathbb{R}^n$ has a basis.
- Let $\mathcal{S}$ be a finite subset of vectors from a nonzero subspace V of $\mathbb{R}^n$ such that $\mathcal{S}$ generates V, i.e., $V = \text{Span } \mathcal{S}$. Then $\mathcal{S}$ contains a basis $\mathcal{S}'$ for V. (**Reduction**)
- Let $\mathcal{S}$ be a set of linearly independent vectors from a nonzero subspace V of $\mathbb{R}^n$. Then V has a basis $\mathcal{S}'$ containing S. (**Extension**)
- Let $\mathcal{S}$ and $\mathcal{S}'$ be two bases for a nonzero subspace V of $\mathbb{R}^n$. Then $\mathcal{S}$ and $\mathcal{S}'$ have the same number of vectors.

Since any two bases for a nonzero subspace have the same number of vectors, we can define the **dimension** of a subspace.

Definition: Let V be a nonzero subspace of $\mathbb{R}^n$ with basis $\mathcal{S} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$. The **dimension** of V is k, denoted $\dim V = k$.

Note:

- $\dim \mathbb{R}^n = n$.
- If $\dim V = k$, then <u>any</u> set of k linearly independent vectors in V is a basis.

1. Consider the matrix $A = \begin{bmatrix} -1 & 1 & 2 & 2 \\ 2 & 0 & -5 & 3 \\ 1 & -1 & -1 & -1 \\ 0 & 1 & -2 & 2 \end{bmatrix}$.

   (a) Find a basis for Col A. What is dim Col A?

   (b) Find a basis for Null A. What is dim Null A?

(c) Show that $\left\{ \begin{bmatrix} 1 \\ 3 \\ -2 \\ 4 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 3 \\ -3 \end{bmatrix}, \begin{bmatrix} -3 \\ 9 \\ 2 \\ 3 \end{bmatrix} \right\}$ is a basis for Col A.

2. Consider the set $\mathcal{S} = \left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ -1 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 5 \\ -3 \end{bmatrix} \right\}$

(a) These vectors live in $\mathbb{R}^n$ for what value of n?

(b) Based on your answer to the previous question, why is $\mathcal{S}$ **NOT** a basis for $\mathbb{R}^n$?

(c) Can $\mathcal{S}$ be extended to a basis for $\mathbb{R}^n$? That is, can you find a basis $\mathcal{S}'$ for $\mathbb{R}^n$ such that $\mathcal{S}'$ contains $\mathcal{S}$? If so, find $\mathcal{S}'$; if not, explain why not.