Section 4.4: Coordinate Systems

In this section, we are going to learn how to represent vectors using different bases. If we write the vector $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, it is understood that we mean the linear combination $\mathbf{v} = 1\mathbf{e}_1 + 2\mathbf{e}_2 + 3\mathbf{e}_3$; that is, it is assumed we are using the **standard basis** for $\mathbb{R}^3$. But, what if we need to use another basis $\mathcal{B}$ for $\mathbb{R}^3$, perhaps due to some physical applications? How would we represent the vector $\mathbf{v}$ then?

Note that we determined the coordinates for $\mathbf{v}$ with respect to the standard basis by writing $\mathbf{v}$ as a linear combination of $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$. If we have a different basis $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ for $\mathbb{R}_3$, we will do the same thing to find the $\mathcal{B}$-coordinates of $\mathbf{v}$, that is, we will write $\mathbf{v}$ as a linear combination of $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$.

Definition: Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$ be a basis for $\mathbb{R}^n$ and let $\mathbf{v}$ be a vector in $\mathbb{R}^n$. Let

$$\mathbf{v} = c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + \cdots + c_n\mathbf{b}_n.$$

The **$\mathcal{B}$-coordinate vector for $\mathbf{v}$** is $[\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$.

It turns out that the scalars $c_1, c_2, \ldots, c_n$ are **unique** and so this definition makes sense.

In what follows, we are going to consider the basis

$$\mathcal{B} = \left\{ \mathbf{b}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 2 \end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix} 1 \\ 1 \\ -2 \\ 1 \end{bmatrix}, \mathbf{b}_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 3 \end{bmatrix}, \mathbf{b}_4 = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 2 \end{bmatrix} \right\}.$$

1. Consider the vector $\mathbf{v} = \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \end{bmatrix}$.

   (a) Write $\mathbf{v}$ as a linear combination of the vectors in $\mathcal{B}$.

   (b) Using your work in the previous question, find the $\mathcal{B}$-coordinate vector for the vector $\mathbf{v}$.

2. Find the $\mathcal{B}$-coordinate vectors for each of the vectors in $\mathcal{B}$, that is, find $[\mathbf{b}_1]_{\mathcal{B}}$, $[\mathbf{b}_2]_{\mathcal{B}}$, $[\mathbf{b}_3]_{\mathcal{B}}$, and $[\mathbf{b}_4]_{\mathcal{B}}$.

Now let's find an easy way to calculate the $\mathcal{B}$-coordinates for a vector $\mathbf{v}$.

3. Suppose $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, \mathbf{b}_4\}$ is a basis for $\mathbb{R}^4$ and $\mathbf{v}$ is a vector in $\mathbb{R}^4$. To find the $\mathcal{B}$-coordinates of $\mathbf{v}$, we start by writing $\mathbf{v}$ as a linear combination of $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, \mathbf{b}_4$, say $\mathbf{v} = c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + c_3\mathbf{b}_3 + c_4\mathbf{b}_4$.

   (a) Write $\mathbf{v} = c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + c_3\mathbf{b}_3 + c_4\mathbf{b}_4$ as a matrix-vector product, i.e., $\mathbf{v} = B\mathbf{w}$ for what matrix B and what vector $\mathbf{w}$?

   (b) Is B invertible? Why or why not?

   (c) Solve $\mathbf{v} = B\mathbf{w}$ for $\mathbf{w}$.

   (d) What <u>**is**</u> $\mathbf{w}$ with respect to $\mathbf{v}$ and the $\mathcal{B}$-coordinates of $\mathbf{v}$?

   (e) Find an expression for $[\mathbf{v}]_{\mathcal{B}}$ in terms of $\mathbf{v}$ and the matrix B.