

### Section 7.3: Basis and Dimension

**Definition:** Let  $V$  be a vector space. A finite set of vectors  $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_k\}$  is **linearly dependent** if there exists scalars  $c_1, c_2, \dots, c_k$ , not all zero, such that  $c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3 + \dots + c_k\mathbf{u}_k = \mathbf{0}$ .

A finite set of vectors  $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_k\}$  is **linearly independent** if the only solution to  $x_1\mathbf{u}_1 + x_2\mathbf{u}_2 + x_3\mathbf{u}_3 + \dots + x_k\mathbf{u}_k = \mathbf{0}$  is  $x_1 = x_2 = \dots = x_k = 0$ .

1. Is the set  $S = \left\{ \mathbf{u}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \mathbf{u}_3 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right\}$  linearly dependent or linearly independent subset of  $\mathbb{R}^3$ ?

2. Based on your work in the previous question, do you think  $S = \{1 + 2x + 3x^2, 4 + 5x + 6x^2, 2 + x\}$  is a linearly dependent or linearly independent subset of  $\mathcal{P}_2$ ? Explain.

3. Is the set  $S = \left\{ \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 4 & 3 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 12 & 9 \\ -3 & 0 \end{bmatrix} \right\}$  linearly dependent or independent? Explain.

**Definition:** Let  $V$  be a vector space. A **basis** for  $V$  is a linearly independent generating set for  $V$ .

It turns out that **every** vector space has a basis (although this is not obvious). However, the basis does not have to have a **finite** number of vectors. For example, the vector space  $\mathcal{P}$  has an infinite basis  $\{1, x, x^2, \dots, x^n, \dots\}$ . For vector spaces with a finite basis, the following theorem holds.

**Theorem** Let  $V$  be a vector space with a finite basis.

- Let  $\mathcal{S}$  be a finite subset of vectors of  $V$  such that  $\mathcal{S}$  generates  $V$ , i.e.,  $V = \text{Span } \mathcal{S}$ . Then  $\mathcal{S}$  contains a basis  $\mathcal{S}'$  for  $V$ . (**Reduction**)
- Let  $\mathcal{S}$  be a set of linearly independent vectors from  $V$ . Then  $V$  has a basis  $\mathcal{S}'$  containing  $\mathcal{S}$ . (**Extension**)
- Let  $\mathcal{S}$  and  $\mathcal{S}'$  be two bases for  $V$ . Then  $\mathcal{S}$  and  $\mathcal{S}'$  have the same number of vectors.

We can now define the **dimension** of a vector space.

**Definition:** Let  $V$  be a vector space with a finite basis  $\mathcal{S} = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ . The **dimension** of  $V$  is  $k$ , denoted  $\dim V = k$ . If  $V$  does not have a finite basis, then we say that  $V$  is **infinite-dimensional**.

**Fact:** If  $V$  is a finite-dimensional vector space, say  $\dim V = k$ , then **any** set of  $k$  linearly independent vectors is a basis for  $V$ .

4. Determine the dimension of the following vector spaces. Describe a basis in each case.

(a)  $\mathcal{P}_2$

(b)  $\mathcal{P}_n$

(c)  $\mathcal{M}_{2 \times 3}$

(d)  $\mathcal{M}_{m \times n}$

(e)  $\mathcal{L}(\mathbb{R}^3, \mathbb{R}^2)$

(f)  $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$

5. Find bases for the following subspaces  $W$  of the given vector space  $V$ . Determine the dimension of those subspaces that are finite-dimensional.

(a)  $W = \left\{ \begin{bmatrix} a & a+b \\ a+b & b \end{bmatrix} \mid a, b \in \mathbb{R} \right\}, V = \mathcal{M}_{2 \times 2}$

(b)  $W = \{p(x) \in \mathcal{P} \mid a_0 = 0\}, V = \mathcal{P}$

(c)  $W = \{p(x) \in \mathcal{P} \mid a_0 = 0\}, V = \mathcal{P}_4$

(d)  $W = \left\{ \begin{bmatrix} a+b & 2a \\ 3a-b & -b \end{bmatrix} \mid a, b \in \mathbb{R} \right\}, V = \mathcal{M}_{2 \times 2}$