Section 3.2: Properties of Determinants

1. Let $A = \begin{bmatrix} 1 & 2 & 1 \\ -1 & 0 & 1 \\ 0 & 3 & 0 \end{bmatrix}$.

   (a) Calculate $\det A$.

   (b) Let B be the matrix obtained by interchanging rows 1 and 2 of A. Calculate $\det B$.

   (c) Let $C = 3A$. Calculate $\det C$.

   (d) Let P be the matrix obtained from A by adding two times row 2 to row 1. Calculate $\det P$.

2. We are now going to investigate the determinants of elementary matrices and how this relates to the previous question.

   (a) Let E_1 be the 3×3 elementary matrix that interchanges rows 1 and 2. What is $\det E_1$? Note that $B = E_1 A$. Determine the relationship between the determinants of B, E_1, and A.

   (b) Let F_1 be the 3×3 elementary matrix that multiplies row 1 by 3, let F_2 be the 3×3 elementary matrix that multiplies row 2 by 3, and let F_3 be the 3×3 elementary matrix that multiplies row 3 by 3. What is $\det F_1$? $\det F_2$? $\det F_3$? Note that $C = F_3 F_2 F_1 A$. Determine the relationship between the determinants of C, F_1, F_2, F_3, and A.

(c) Let E_2 be the 3×3 elementary matrix that adds two times rows 2 to row 1. What is $\det E_2$? Note that $P = E_2 A$. Determine the relationship between the determinants of P, E_2, and A.

3. Fill in the blanks below.

Theorem Let A be an $n \times n$ matrix. Then

- If B is obtained from A be interchanging two rows of A, then $\det B =$ ______________.
- If $B = rA$, then $\det B =$ ______________.
- If B is obtained from A by adding a multiple of some row to another row, then $\det B =$ ______________.
- For any elementary matrix E, we have $\det EA =$ ______________.

4. Now we want to determine the determinant of an invertible matrix and the relationship between the determinant of a matrix and its inverse.

(a) Suppose A is an invertible $n \times n$ matrix.
Then A is a product of ______________ ______________,
say $A =$ ____________________________ where each _______ is an ______________ ______________.
Now $\det E_i \neq$ _______ and $\det A =$ ______________________________________
so that $\det A \neq$ _______.

Now suppose A is an $n \times n$ matrix that is NOT invertible. Let $R = \text{rref}(A)$.
Since A is not invertible, R must have a ______________ row.
Thus, $\det R =$ _______. Now we need to relate $\det R$ and $\det A$.
First, there exists an ______________ ______________ P such that $R =$ __________
and $\det R =$ __________ __________.
Since P is invertible, $\det P \neq$ _______ so since $\det R =$ _______
and $\det R =$ __________ __________, it must be that $\det A =$ _______.

(b) Now let's determine the relationship between $\det A$ and $\det(A^{-1})$
Let A be an invertible matrix. So, A is a product of elementary matrices and A^{-1} exists and is also a product of elementary matrices.
Thus $\det(AA^{-1}) = \det$ _______ $\det$ _______. Since $AA^{-1} = I$ and $\det I =$ _______, so

$$
\begin{aligned}
\underline{\hspace{2em}} &= \det I \\
&= \det(AA^{-1}) \\
&= \det \underline{\hspace{2em}} \det \underline{\hspace{2em}}
\end{aligned}
$$

Thus, $1 = \det$ _______ $\det$ _______ so that $\det(A^{-1}) = 1/($______________$)$.

(c) **Conclusion:** Let A be an $n \times n$ matrix.

- A is ________________ if and only if $\det A \neq$ __________.
- If A is invertible, then $\det(A^{-1}) = 1/($_____________$)$.

5. Let A_n denote the following $n \times n$ matrix.

$$A_n = \begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 2 & 2 & \cdots & 2 \\ 1 & 2 & 3 & 3 & \cdots & 3 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 2 & 3 & 4 & \cdots & n \end{bmatrix}$$

(a) Determine A_3, A_4, and A_5. Use elementary row operations and properties of determinants to compute the determinants of these matrices.

(b) Base on your work in part (a), use elementary row operations and properties of determinants to compute $\det A_n$, the determinant of the matrix A_n for an integer $n \geq 3$.