Section 3.1: Calculating Determinants

Definitions: Let A be an $n \times n$ matrix.

- The notation A_{ij} means the $(n-1) \times (n-1)$ matrix obtained from A be deleting the ith row and the jth column.
- The <u>(i, j)-cofactor</u> $c_{ij} = (-1)^{i+j} \det A_{ij}$.
- $\det A = a_{11}c_{11} + a_{12}c_{12} + \cdots + a_{1n}c_{1n}$.

We call this the "cofactor expansion," and the determinant is defined by cofactor expansion along the first row. It turns out that the determinant of a matrix can be calculated by cofactor expansion along <u>ANY</u> row or <u>ANY</u> column.

Theorem Let A be an $n \times n$ matrix. Then

- $\det A = a_{i1}c_{i1} + a_{i2}c_{i2} + \cdots a_{in}c_{in}$ for $i = 1, 2, \ldots, n$ (cofactor expansion along the ith row), and
- $\det A = a_{1j}c_{1j} + a_{2j}c_{2j} + \cdots a_{nj}c_{nj}$ for $j = 1, 2, \ldots, n$ (cofactor expansion along the jth column).

1. Consider the matrix $A = \begin{bmatrix} 1 & 2 & 1 & -1 \\ 0 & -1 & 0 & 1 \\ 4 & -3 & 2 & -1 \\ 0 & 3 & 0 & -2 \end{bmatrix}$.

   (a) Find the determinant of A by cofactor expansion along the second row.

   (b) Find the determinant of A by cofactor expansion along the third column.

2. Now consider the matrix $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -6 & -1 \\ 0 & 0 & -3 & 2 \\ 0 & 0 & 0 & 4 \end{bmatrix}$.

   (a) Which row or column would you use to calculate the determinant? Why?

   (b) Calculate the determinant along the row or column you specified in the previous question.

3. The matrix in the previous question is called upper triangular. Why was it so easy to calculate its determinant? What is the determinant of an upper triangular matrix?

4. What do you think a lower triangular matrix looks like? Do you think it would be easy to calculate the determinant of a lower triangular matrix? What is the determinant of a lower triangular matrix?