Section 5.3: Diagonalization

1. Let $A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}$.

   (a) Determine the eigenvalues of A.

   (b) Find a basis for each eigenspace.

   (c) Form matrix P from the bases vectors you found in the previous question. Determine $P^{-1}AP$. What do you notice about $P^{-1}AP$?

What happened in the previous question is very special, that is, the matrix A is **similar** to a diagonal matrix. We will see that this quality, being similar to a diagonal matrix, is related to the number of linearly independent eigenvectors the matrix A has.

Definition: An $n \times n$ matrix A is **diagonalizable** if $A = PDP^{-1}$ for some diagonal matrix D and invertible matrix P.

Thus, the matrix A in Question 1 is **diagonalizable**.

2. Let $A = \begin{bmatrix} 2 & 4 & 3 \\ -4 & -6 & -3 \\ 3 & 3 & 1 \end{bmatrix}$.

   (a) Determine the eigenvalues of A.

   (b) Find a basis for each eigenspace.

   (c) Is the matrix A diagonalizable? Why or why not?

3. Let A be an $n \times n$ matrix. Fill in blanks:

   (a) A is diagonalizable if and only if A has ____ linearly ______________ ______________.

   (b) $A = PDP^{-1}$ for some diagonal matrix D and invertible matrix P if and only if the columns of P are ____ linearly ______________ ______________ for A and the diagonal entries of D are the corresponding ______________.

4. Suppose $A = PDP^{-1}$ where $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$, $P = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix}$, and $D = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. What are the eigenvalues of A? Determine the dimension of each corresponding eigenspace.

5. Compute A^8 where $A = \begin{bmatrix} 4 & -3 \\ 2 & -1 \end{bmatrix}$. (**Hint:** There is an easy way to do this and a hard way.)

6. Let A be a 4×4 matrix with eigenvalues $5, 3$ and -2, and suppose you know that the eigenspace for $\lambda = 3$ has dimension 2. Is A diagonalizable? Why or why not?

7. Let A be an $n \times n$ matrix with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$ and corresponding eigenspaces $E_1, E_2, \ldots, E_k$ (i.e., $E_i = \text{Null }(A - \lambda_i I)$). Fill in blanks:

   (a) A is diagonalizable if and only if dim E_1 + dim $E_2 + \cdots +$ dim $E_k =$ ____.

   (b) A is diagonalizable if and only if dim E_i equals the ______________ of λ_i for each $i = 1, 2, \ldots, k$.

   (c) Suppose $\mathcal{B}_i$ is a basis for E_i for each $i = 1, 2, \ldots, k$. Then A is diagonalizable if and only if $\mathcal{B}_1 \cup \mathcal{B}_1 \cup \cdots \cup \mathcal{B}_k$ (the total collection of the vectors in the sets $\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_k$) forms a basis for ______________.

8. Let A be an $n \times n$ matrix. Answer each of the following **TRUE** or **FALSE** and justify your answer:

   (a) If $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are eigenvectors of A corresponding to **distinct** eigenvalues of A, then $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is linearly independent.

   (b) If A has n distinct eigenvalues, then A is diagonalizable.

   (c) If A is diagonalizable, then A has n distinct eigenvalues.

9. A 5×5 matrix has eigenvalue -3 with multiplicity 4 and eigenvalue 7 with multiplicity 1.

   (a) Under what conditions is the matrix diagonalizable? Explain.

   (b) Under what conditions is the matrix not diagonalizable? Explain.

10. Let A be a 4×4 matrix with eigenvalues 2 and 7. Let E_1 be the eigenspace for eigenvalue 2 and let E_2 be the eigenspace for eigenvalue 7. In each question below, write the characteristic polynomial of A, or explain why there is insufficient information to determine the characteristic polynomial.

    (a) dim $E_1 = 3$

    (b) dim $E_2 = 2$

    (c) A is diagonalizable and dim $E_2 = 2$.