Math 175
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Section 5.2: The Characteristic Polynomial

In this handout, we are going to find the eigenvalues of given matrix A.

1. Let $A = \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix}$.

   (a) Let t be a variable. Find $A - tI$.

   (b) Determine $\det(A - tI)$.

   (c) Solve $\det(A - tI) = 0$.

   (d) You should have found $t = -7$ and $t = 3$ as solutions to the previous question. Verify that -7 and 3 are eigenvalues by finding basis for the associated eigenspaces. (**Recall** that the eigenspace for an eigenvalue λ is Null $(A - \lambda I)$.)

2. Based on your work in the previous question, write a few sentences about how to find eigenvalues of an $n \times n$ matrix A.

3. Fill in blanks:

   (a) If t is an eigenvalue, then $A\mathbf{x} = t\mathbf{x}$ has _____, or _____ = ____ has _____ solutions.

   (b) If there are nonzero solutions to $(A - tI)\mathbf{x} = \mathbf{0}$, then $A - tI$ is _____ invertible so $\det(A - tI) =$ ___ .

Definition: Let A be an $n \times n$ matrix A. The equation $\det(A - tI) = 0$ is called the **characteristic equation** and the polynomial $\det(A - tI)$ is called the **characteristic polynomial** of A.

4. Consider the linear transformation $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} -3x_1 \\ -8x_1 + x_2 \\ -12x_1 + x_3 \end{bmatrix}$.

   (a) Determine the standard matrix A for T.

   (b) Determine the characteristic polynomial for A.

   (c) Determine the eigenvalues of T.

   (d) Determine a basis each eigenspace.

5. In the previous question, what do you notice about the matrix A and its eigenvalues? What are the eigenvalues of an upper triangular matrix? A lower triangular matrix? Why is this?

Example: Suppose A is a matrix such that its characteristic polynomial $\det(A - tI)$ is $(t-5)^2(t+6)(t-7)^3(t-8)^4$. Then, we say that eigenvalue 5 has **multiplicity** 2, eigenvalue -6 has **multiplicity** 1, eigenvalue -7 has **multiplicity** 3, and eigenvalue 8 has **multiplicity** 4.

6. Let $A = \begin{bmatrix} 5 & -2 & 6 & -1 \\ 0 & 3 & -8 & 0 \\ 0 & 0 & 5 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

   (a) Determine the eigenvalues of A and their multiplicities.

   (b) Determine basis for each eigenspace. Compare the multiplicity of an eigenvalue with the dimension of its corresponding eigenspace. What do you notice?

7. Suppose A is a 5×5 matrix with three eigenvalues: 4, 6, and 7. Let E_1, E_2, E_3 be the eigenspaces corresponding to eigenvalues 4, 6, 7, respectively. Write all the possible characteristic polynomials of A that are consistent with the following information:

   (a) $\dim E_2 = 3$

   (b) $\dim E_1 = 2$

   (c) $\dim E_1 = 1$ and $\dim E_2 = 2$

   (d) $\dim E_2 = 2$ and $\dim E_3 = 2$

8. Prove that if A is not invertible, then $\lambda = 0$ is an eigenvalue.