Section 5.1: Eigenvalues and Eigenvectors

1. Let $A = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}$, $\mathbf{u} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, and $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Determine $A\mathbf{u}$ and $A\mathbf{v}$. Draw $\mathbf{u}, \mathbf{v}, A\mathbf{u}, A\mathbf{v}$ in the plane. What do you notice?

You might have noticed that $A\mathbf{v} = 2\mathbf{v}$. When this happens, that is, $A\mathbf{v} = \lambda\mathbf{v}$, we call λ an **eigenvalue** and $\mathbf{v}$ is an **eigenvector**.

Definitions: Let A be an $n \times n$ matrix. A nonzero vector $\mathbf{v} \in \mathbb{R}^n$ is an **eigenvector** if $A\mathbf{v} = \lambda\mathbf{v}$ for some $\lambda \in \mathbb{R}$. We call λ the **eigenvalue** of A that corresponds to $\mathbf{v}$.

2. Consider the matrix $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$.

   (a) Let $\mathbf{u} = \begin{bmatrix} 6 \\ -5 \end{bmatrix}$, and $\mathbf{v} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$. Are either of these vectors eigenvectors of A?

   (b) Is 7 an eigenvalue of A? (**Hint:** If so, then there is a solution to $A\mathbf{x} = 7\mathbf{x}$, or $(A - 7I)\mathbf{x} = \mathbf{0}$. Can you solve this equation?

We want to learn how to find eigenvalues and eigenvectors. It turns out that it is easier to determine the **eigenvalues** and then find the corresponding **eigenvectors**.

In general, λ is an eigenvalue of $(A - \lambda I)\mathbf{x} = \mathbf{0}$ has nonzero solutions.

Definition: Let λ be an eigenvalue of an $n \times n$ matrix A. The **eigenspace of A corresponding to eigenvalue** λ is the set of solutions to $(A - \lambda I)\mathbf{x} = \mathbf{0}$ or Null $(A - \lambda I)$.

Clearly, the eigenspace for eigenvalue λ is a subspace (since it's the null space of $A - \lambda I$).

3. Let $A = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}$. It turns out that 2 is an eigenvalue. Find a basis for the corresponding eigenspace.

4. Let A be an $n \times n$ matrix with eignevalue λ.

   (a) Let $\mathbf{u}, \mathbf{w}$ be eigenvectors for eigenvalue λ. Prove $\mathbf{u} + \mathbf{w}$ is an eigenvector for eigenvalue λ.

   (b) Let $\mathbf{u}$ be an eigenvector for eigenvalue λ and let c be a scalar. Prove $c\mathbf{u}$ is an eigenvector for eigenvalue λ.