

### Exam 1

106 points possible. 100 points maximum. Throughout this exam,  $\mathbb{R}$  denotes the set of real numbers.

1. (18 pts.) Answer each of the following by CIRCLING True or False. No explanation necessary.

- (a) **True** or **False**: If  $A$  is an  $m \times n$  matrix and  $\mathbf{u}$  is an  $n \times 1$  vector, then  $A\mathbf{u}$  is a linear combination of the rows of  $A$ .
- (b) **True** or **False**: Let  $\mathbf{u}, \mathbf{v} \in \mathbb{R}^2$ . Then every vector in  $\mathbb{R}^2$  can be written as a linear combination of  $\mathbf{u}$  and  $\mathbf{v}$ .
- (c) **True** or **False**: If the reduced row echelon form of an augmented matrix for a system of linear equations contains a zero row, then the system is consistent.
- (d) **True** or **False**: If the last row in the reduced row echelon form of an augmented matrix for a system of linear equations contains a nonzero entry, then the system is inconsistent.
- (e) **True** or **False**: If  $A$  is an  $m \times n$  matrix with rank  $m$ , then the span of the columns of  $A$  is  $\mathbb{R}^m$ .
- (f) **True** or **False**: If  $S_1$  and  $S_2$  are two subsets of  $\mathbb{R}^n$  with  $\text{Span } S_1 = \text{Span } S_2$ , then  $S_1 = S_2$ .

2. (12 pts.) Let  $A$  and  $B$  be  $m \times n$  matrices and let  $\mathbf{u}$  be a  $n \times 1$  vector. Without using Theorem 1.2, prove that  $(A + B)\mathbf{u} = A\mathbf{u} + B\mathbf{u}$ .

3. (24 pts.) Give an example of each of the following or state that no such example exists.
- (a)  $2 \times 2$  matrices  $A$  and  $B$  and a  $2 \times 1$  nonzero vector  $\mathbf{u}$  such that  $A\mathbf{u} = B\mathbf{u}$  yet  $A \neq B$  and neither  $A$  nor  $B$  is the zero matrix.
  
  
  
  
  
  
  
  
  
  
  - (b) A  $3 \times 2$  matrix  $A$  such that the equation  $A\mathbf{x} = \mathbf{b}$  has at least one solution for each  $\mathbf{b} \in \mathbb{R}^3$ .
  
  
  
  
  
  
  
  
  
  
  - (c) A set  $S$  of vectors in  $\mathbb{R}^3$  such that  $\text{Span } S = \mathbb{R}^3$ .
  
  
  
  
  
  
  
  
  
  
  - (d) A  $3 \times 4$  augmented matrix in reduced row echelon form so that the corresponding system of equations has infinitely many solutions.
  
  
  
  
  
  
  
  
  
  
  - (e) A  $3 \times 4$  augmented matrix in reduced row echelon form so that the corresponding system of equations has exactly one solution.
  
  
  
  
  
  
  
  
  
  
  - (f) A  $3 \times 4$  augmented matrix in reduced row echelon form so that the corresponding system of equations has no solution.

4. (20 pts.) Find the general solution in vector form to

$$\begin{aligned}x_1 - x_2 - 3x_3 + x_4 - x_5 &= -2 \\-2x_1 + 2x_2 + 6x_3 - 6x_5 &= -6 \\3x_1 - 2x_2 - 8x_3 + 3x_4 - 5x_5 &= -7\end{aligned}$$

using the Gaussian elimination algorithm, i.e., by finding the reduced row echelon form of a certain matrix. (**Note:** This means you are allowed to use your calculator only as a check.)

5. (10 pts.) Let  $\mathbf{u}$  and  $\mathbf{v}$  be solutions to  $A\mathbf{x} = \mathbf{0}$ , where  $A$  is an  $m \times n$  matrix. Prove that  $\mathbf{u} + \mathbf{v}$  is a solution to  $A\mathbf{x} = \mathbf{0}$ .

6. (10 pts.) Find a subset of the set  $S$  below that is as small as possible with the same span as  $S$ .

$$S = \left\{ \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ -7 \\ 2 \end{bmatrix}, \begin{bmatrix} -5 \\ 7 \\ 6 \end{bmatrix} \right\}$$

7. (12 pts.) Find an equation relating  $a, b$  and  $c$  so that the linear system

$$\begin{aligned} x_1 - 4x_2 + 7x_3 &= a \\ 3x_2 - 5x_3 &= b \\ -2x_1 + 5x_2 - 9x_3 &= c \end{aligned}$$

is consistent.