Name: ________________________

Math 175, Spring 2008
H. Jordon

Exam 4

100 points maximum. Throughout this exam, $\mathbb{R}$ denotes the set of real numbers.

1. (12 pts.) Answer each of the following by CIRCLING True or False. No explanation necessary.

   (a) **True** or **False**: Let A be an invertible $n \times n$ matrix with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$. Then the eigenvalues of A^{-1} are $1/\lambda_1, 1/\lambda_2, \ldots, 1/\lambda_k$

   (b) **True** or **False**: If A is an $n \times n$ matrix with n distinct eigenvectors, then A is a diagonalizable.

   (c) **True** or **False**: Every vector space has a finite basis.

   (d) **True** or **False**: The set of polynomials of odd degree is a subspace of $\mathcal{P}$.

2. (10 pts.) Let $V = \{(x, y) \mid x, y \in \mathbb{R}\}$ and define the operations of vector addition $\oplus$ and scalar multiplication $\odot$ as follows:

$$(a, b) \oplus (c, d) = (a + c + 2, b + d)$$

   and for a real number k,

$$k \odot (a, b) = (ka, b).$$

   Determine whether scalar multiplication distributes over scalar addition, i.e., whether

$$(k + \ell) \odot u = (k \odot u) \oplus (\ell \odot u)$$

   where k and ℓ are scalars and u is a vector in V.

3. (18 pts.) Let $A = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 3 & 2 \\ 1 & 1 & 2 \end{bmatrix}$. The eigenvalues of A are 1 and 5.

(a) Determine the matrix whose determinant is the characteristic polynomial for A. (**Note:** You do not have to evaluate the determinant.)

(b) Explain, in a short sentence, how the eigenvalues of the matrix A are related to the determinant of the matrix given in part (a).

(c) Explain, in a short sentence, how the nullspace of the matrix $A - 5I$ consists of the eigenvectors of A for eigenvalue 5.

(d) What is the dimension of the nullspace of the matrix $A-5I$? Find a basis for Null($A-5I$).

(e) What is the dimension of the nullspace of the matrix $A-I$? Find a basis for Null($A-I$).

(f) Is A diagonalizable or not? If so, write A as PDP^{-1} for matrices P and D with D a diagonal matrix. If not, explain why not.

4. (20 pts.) Determine, with explanation, whether or not the set W is a subspace of the given vector space V. If W is a subspace, determine a basis for W and its dimension.

(a) $W = \left\{ \begin{bmatrix} a & a+2b \\ a-3b & b \end{bmatrix} \mid a, b \in \mathbb{R} \right\}$, $V = \mathcal{M}_{2\times 2}$

(b) $W = \{a_2x^2 + a_1x + a_0 \mid a_2 + a_1 + a_0 = 0\}$, $V = \mathcal{P}_2$

5. (10 pts.) Prove that if A is not invertible, then 0 is an eigenvalue of A.

6. (10 pts.) Consider the set $S = \{x^2 - 3x + 2, 2x^2 - x, x^2 + rx - 2\}$.

(a) Determine, if possible, value(s) of r which would make the set S linearly dependent.

(b) Determine, if possible, value(s) of r which would make the set S linearly independent.

7. (20 pts.) Consider the function $T : \mathcal{M}_{2\times 2} \to \mathbb{R}^4$ defined by $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a+b \\ 2b \\ c+d \\ 2d \end{bmatrix}$.

(a) Prove that T is linear.

(b) Determine, with explanation, whether or not T is one-to-one.

(c) Determine, with explanation, whether or not T is onto.