Math 175, Fall 2010
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Solutions to Take-Home Exam 4

1. **Let $V = \{(a,b) \mid a,b \in \mathbb{R}\}$ and define the operations of vector addition $\oplus$ and scalar multiplication $\odot$ as follows:**

$$(a,b) \oplus (c,d) = (a+c+1, b+d-1)$$

and for a real number k,

$$k \odot (a,b) = (a, kb).$$

It turns out that $(V, \oplus, \odot)$ is not a vector space. However, $(V, \oplus, \odot)$ does satisfy many of the axioms of a vector space. Determine, with explanation, which of the 8 axioms of a vector space, giving on pp. 490 of your textbook, $(V, \oplus, \odot)$ does satisfy. If $(V, \oplus, \odot)$ does not satisfy an axiom, give a specific example to show the given axiom fails.

(1) $(a,b) \oplus (c,d) = (a+c+1, b+d-1) = (c+a+1, d+b-1) = (c,d) \oplus (a,b)$

(2) $(a,b) \oplus [(c,d) \oplus (e,f)] = (a,b) \oplus (c+e+1, d+f-1) = (a+(c+e+1)+1, b+(d+f-1)-1) = (a+c+e+2, b+d+f-2)$ and $[(a,b) \oplus (c,d)] \oplus (e,f) = (a+c+1, b+d-1) \oplus (e,f) = ((a+c+1)+e+1, (b+d-1)+f-1) = (a+c+e+2, b+d+f-2)$

(3) $0_V = (-1,1) \in V$ since $(a,b) \oplus (-1,1) = (a-1+1, b+1-1) = (a,b)$

(4) Let $(a,b) \in V$. Consider $(-2-a, 2-b) \in V$ and $(a,b) \oplus (-2-a, 2-b) = (a+(-2-a)+1, b+(2,b)-1) = (-1,1) = 0_V$.

(5) $1 \odot (a,b) = (a, 1 \cdot b) = (a,b)$

(6) $(st) \odot (a,b) = (a, (st)b) = (a, s(tb)) = s \odot (a, tb) = s \odot [(t \odot (a,b)]$

(7) Consider $2 \odot [(1,2) \oplus (3,4)] = 2 \odot (5,5) = (5,10)$ and $2 \odot (1,2) \oplus 2 \odot (3,4) = (1,4) \oplus (3,8) = (5,11)$ so that $2 \odot [(1,2) \oplus (3,4)] \neq 2 \odot (1,2) \oplus 2 \odot (3,4)$.

(8) Consider $(1+2) \odot (3,4) = 3 \odot (3,4) = (3,12)$ and $1 \odot (3,4) \oplus 2 \odot (3,4) = (3,4) \oplus (3,8) = (7,11)$ so that $(1+2) \odot (3,4) \neq 1 \odot (3,4) \oplus 2 \odot (3,4)$.

2. **Let A be an $n \times n$ matrix with eigenvalue λ.**

   (a) **Let k be a positive integer. Prove that λ^k is an eigenvalue of A^k.**

   Let A be an $n \times n$ matrix with eigenvalue λ. Then, there exists a nonzero vector $\mathbf{v}$ such that $A\mathbf{v} = \lambda\mathbf{v}$. Consider

$$
\begin{aligned}
A^k\mathbf{v} &= A^{k-1}(A\mathbf{v}) \\
&= A^{k-1}(\lambda\mathbf{v}) \\
&= \lambda(A^{k-1}\mathbf{v}) \\
&= \lambda[A^{k-2}(A\mathbf{v})] \\
&= \lambda[A^{k-2}(\lambda\mathbf{v})] \\
&= \lambda^2(A^{k-2}\mathbf{v}) \\
&= \vdots \\
&= \lambda^k\mathbf{v}
\end{aligned}
$$

   Thus, λ^k is an eigenvalue of A^k.

   (b) **Suppose there exists a positive integer k such that $A^k = I_n$. Determine the eigenvalues of A.**

   Suppose $A^k = I$. We have already noted that if λ is an eigenvalue of A, then λ^k is an eigenvalue of A and furthermore, if $\mathbf{v}$ is the corresponding eigenvector for eigenvalue λ, then $\mathbf{v}$ is also an eigenvector for eigenvalue λ^k. So,

$$
\begin{aligned}
A^k\mathbf{v} &= \lambda^k\mathbf{v} \\
I\mathbf{v} &= \lambda^k\mathbf{v} \\
\mathbf{v} &= \lambda^k\mathbf{v}.
\end{aligned}
$$

   Note that if v_i is the ith coordinate of $\mathbf{v}$, then, $v_i = \lambda^k v_i$ or $1 = \lambda^k$. Thus, λ is the kth primitive root of unity, given by $\lambda = [\cos(2\pi/k) + i\sin(2\pi/k)]^j$ where j is relatively prime to k.

3. **Let a, b be distinct real numbers. Prove that** $\begin{bmatrix} a & 0 & 0 \\ 1 & a & 0 \\ 0 & 0 & b \end{bmatrix}$ **is not diagonalizable.**

Clearly, the eigenvalues of A are a, with multiplicity 2, and b with multiplicity 1. We wish to show that the eigenspace for eigenvalue a has dimension 1. Consider $A - aI = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & b-a \end{bmatrix}$, whose rref is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. Thus, the dimension of $\text{Null}(A-aI)$ is 1 and A is not diagonalizable.

4. **Let $T : \mathcal{M}_{2\times 2} \to \mathcal{P}_3$ be defined by** $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = (b-3d) + (2a+c-d)x + (a-2b+4d)x^2 + (5b+c)x^3$.

   (a) **Prove that T is a linear transformation.**

   Let $\begin{bmatrix} a & b \\ c & d \end{bmatrix}, \begin{bmatrix} e & f \\ g & g \end{bmatrix} \in \mathcal{M}_{2\times 2}$ and s be a scalar. Then

$$\begin{aligned}
&T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} e & f \\ g & h \end{bmatrix}\right) \\
&= T\left(\begin{bmatrix} a+e & b+f \\ c+g & d+h \end{bmatrix}\right) \\
&= [(b+f)-3(d+h)] + [2(a+e)+(c+g)-(d+h)]x + [(a+e)-2(b+f)+4(d+h)]x^2 \\
&\quad +[5(b+f)+(c+g)]x^3 \\
&= [(b-3d)+(2a+c-d)x+(a-2b+4d)x^2+(5b+c)x^3] \\
&\quad +[(f-3h)+(2e+g-h)x+(e-2f+4h)x^2+(5f+g)x^3] \\
&= T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) + T\left(\begin{bmatrix} e & f \\ g & h \end{bmatrix}\right),
\end{aligned}$$

   and

$$\begin{aligned}
T\left(s\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) &= T\left(\begin{bmatrix} sa & sb \\ sc & sd \end{bmatrix}\right) \\
&= (sb-3sd)+(2sa+sc-sd)x+(sa-2sb+4sd)x^2+(5sb+sc)x^3 \\
&= s[(b-3d)+(2a+c-d)x+(a-2b+4d)x^2+(5b+c)x^3] \\
&= sT\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right).
\end{aligned}$$

(b) **Find a basis for the null space of T. Is T one-to-one? Why or why not?**

We solve $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = (b-3d) + (2a+c-d)x + (a-2b+4d)x^2 + (5b+c)x^3 = 0 + 0x + 0x^2 + 0x^3$. So,

$$\begin{aligned} b - 3d &= 0 \\ 2a + c - d &= 0 \\ a - 2b + 4d &= 0 \\ 5b + c &= 0 \end{aligned}$$

Consider the rref$\left(\begin{bmatrix} 0 & 1 & 0 & -3 \\ 2 & 0 & 1 & -1 \\ 1 & -2 & 0 & 4 \\ 0 & 5 & 1 & 0 \end{bmatrix}\right) = I_4$. Hence, the only solution is $a = b = c = d = 0$ so that $\text{Null}(T) = 0_{\mathcal{M}_{2\times 2}}$. Hence, T is one-to-one.

(c) **Find a spanning set for the range of T. Is T onto? Why or why not?**

Range $T = \text{Span}\{2x + x^2, 1 - 2x^2 + 5x^3, x + x^3, -3 - x + 4x^2\}$. T is onto since dim Null $T = 0$ and dim $\mathcal{M}_{2\times 2} = 4$ so that dim Range $T = 4 =$ dim $\mathcal{P}_3$.

5. **Let $(V, \oplus, \odot)$ be a vector space. Prove that $(-c) \odot (-v) = c \odot v$ for every vector v in V and scalar c using the axioms and properties of a vector space (Theorem 7.2). Justify each of your steps by stating which axiom/property you are using.**

Let $(V, \oplus, \odot)$ be a vector space. Let c a scalar and let $v \in V$. Consider

$$\begin{aligned} (-c) \odot (-v) &= (-c) \odot [(-1) \odot v] \quad \text{Thm 7.2 (g)} \\ &= (-c \cdot -1) \odot v \quad \text{Axiom 6} \\ &= c \odot v. \end{aligned}$$

6. **Let A be a $n \times n$ diagonalizable matrix. Prove that A^T is diagonalizable. What are the eigenvalues of A^T?**

Let A be a $n \times n$ diagonalizable matrix. Then, there exists an invertible matrix P and diagonal matrix D such that $A = PDP^{-1}$. Then, $A^T = (PDP^{-1})^T = (P^{-1})^T D^T P^T = (P^T)^{-1} D^T P^T$. Let $Q = (P^T)^{-1}$ and note that $D^T = D$ since D is a diagonal matrix. Then Q is invertible and $A^T = QDQ^{-1}$. Hence, the entries on the diagonal of D are the eigenvalues of A^T; hence A and A^T have the same eigenvalues.

7. **Let** $W = \left\{ \begin{bmatrix} a-b & b-c \\ c-a & b \end{bmatrix} \mid a, b, c \in \mathbb{R} \right\}$**.**

(a) **Show that** W **is a subspace of** $V = \mathcal{M}_{2\times 2}$**.**

$W = \text{Span}\left\{ \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right\}$ so that W is a subspace.

(b) **Find a basis for** W **and determine its dimension.**

Consider $\text{rref}\left(\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ so the spanning set above is linearly independent. Thus, $\dim W = 3$ and this spanning set is a basis.

8. **Find the value(s) of** r **so that the matrix** $\begin{bmatrix} -1 & 1 \\ r & -1 \end{bmatrix}$ **is in the span of the set** $S = \left\{ \begin{bmatrix} -1 & 2 \\ -1 & 1 \end{bmatrix}, \begin{bmatrix} 2 & -1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 2 \\ 0 & 3 \end{bmatrix} \right\}$**.**

Solve $c_1 \begin{bmatrix} -1 & 2 \\ -1 & 1 \end{bmatrix} + c_2 \begin{bmatrix} 2 & -1 \\ -1 & 0 \end{bmatrix} + c_3 \begin{bmatrix} -1 & 2 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ r & -1 \end{bmatrix}$, giving the following equations:

$$\begin{aligned} -c_1 + 2c_2 - c_3 &= -1 \\ 2c_1 - c_2 + 2c_3 &= 1 \\ -c_1 - c_2 &= r \\ c_1 + 3c_3 = -1. \end{aligned}$$

The solution to this system is $c_1 = 1, c_2 = -1/3, c_3 = -2/3$ so that $r = -2/3$.

9. **Determine, with explanation, whether the set** $S = \{-1 + 2x^2 + x^3, 1 + x - x^2 - x^3, 2x - x^2 + x^3, -1 + 3x + x^2 + 2x^3\}$ **is linearly independent.**

We wish to solve $c_1(-1+2x^2+x^3)+c_2(1+x-x^2-x^3)+c_3(2x-x^2+x^3)+c_4(-1+3x+x^2+2x^3) = 0+0x+0x^2+0x^3$, or $(-c_1+c_2-c_4)+(c_2+2c_3+3c_4)x+(2c_1-c_2-c_3+c_4)x^2+(c_1-c_2+c_3+2c_4)x^3 = 0 + 0x + 0x^2 + 0x^3$. Equating coefficients on both sides gives rise to the following equations:

$$\begin{aligned} -c_1 + c_2 - c_4 &= 0 \\ c_2 + 2c_3 + 3c_4 &= 0 \\ 2c_1 - c_2 - c_3 + c_4 &= 0 \\ c_1 - c_2 + c_3 + 2c_4 &= 0 \end{aligned}$$

Since $\text{rref}\left(\begin{bmatrix} -1 & 1 & 0 & -1 \\ 0 & 1 & 2 & 3 \\ 2 & -1 & -1 & 1 \\ 1 & -1 & 1 & 2 \end{bmatrix} \right) = I_4$, the only solution to this system is $c_1 = c_2 = c_3 = c_4 = 0$ so that these vectors are linearly independent.