Section 1.7

Definition: A set S of vectors $S = \{\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3}, \ldots, \mathbf{u_k}\}$ is **linearly dependent** if there exists scalars $c_1, c_2, \ldots, c_k$, not all zero, such that $c_1\mathbf{u_1} + c_2\mathbf{u_2} + c_3\mathbf{u_3} + \cdots + c_k\mathbf{u_k} = \mathbf{0}$.

A set S of vectors $S = \{\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3}, \ldots, \mathbf{u_k}\}$ is **linearly independent** if the only solution to $x_1\mathbf{u_1} + x_2\mathbf{u_2} + x_3\mathbf{u_3} + \cdots + x_k\mathbf{u_k} = \mathbf{0}$ is $x_1 = x_2 = \cdots = x_k = 0$.

1. Let $\mathbf{u_1} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\mathbf{u_2} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$, and $\mathbf{u_3} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$. We want to determine whether or not $\{\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3}\}$ is a linearly dependent set or not.

   (a) Write out the equations determined by $x_1\mathbf{u_1} + x_2\mathbf{u_2} + x_3\mathbf{u_3} = \mathbf{0}$.

   (b) Set up an augmented matrix $[A|\ \mathbf{0}]$ for your equations.

   (c) Find the rref. Solve for x_1, x_2, x_3.

   (d) Are $\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3}$ linearly independent or linearly dependent? Justify your answer.

2. Let $A = \begin{bmatrix} 2 & -8 & 1 \\ 0 & 5 & 4 \\ 0 & 0 & -3 \end{bmatrix}$.

(a) Are the columns of A linearly independent or linearly dependent? Justify your answer.

(b) Do the columns of A span $\mathbb{R}^3$? Why or why not?

3. Answer each of the following TRUE or FALSE. Provide a short justification in each case.

(a) A set of vectors containing the zero vector is always linearly dependent.

(b) If A is an $m \times n$ matrix such that $A\mathbf{x} = \mathbf{0}$ has infinitely many solutions, then the columns of A are linearly independent.

(c) If A is an $m \times n$ matrix such that the rref(A) has at least one column without a pivot, then the columns of A are linearly dependent.