Section 4.5: Matrix Representations of Linear Operators

In this section, we are going to apply the change of basis ideas from the last section to linear operator $T : \mathbb{R}^n \to \mathbb{R}^n$. (Since $n = m$ for this linear transformation, we call T a **linear operator**.)

Note that we determined the coordinates for $\mathbf{v}$ with respect to the basis $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$ by first writing $\mathbf{v}$ as a linear combination of $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n$, say $\mathbf{v} = c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + \cdots + c_n\mathbf{b}_n$. Then, **$\mathcal{B}$-coordinate vector for $\mathbf{v}$** is $[\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$. Recall also that if B is the $n \times n$ matrix $B = [\mathbf{b}_1\ \mathbf{b}_2\ \ldots\ \mathbf{b}_n]$, then

$$[\mathbf{v}]_{\mathcal{B}} = B^{-1}\mathbf{v}.$$

Recall that for every linear transformation T, there is a corresponding matrix A such that $T(\mathbf{v}) = A\mathbf{v}$. We called A the **standard** matrix for T because it comes from the standard vectors. If $T : \mathbb{R}^n \to \mathbb{R}^n$, then A is the $n \times n$ matrix $A = [T(\mathbf{e}_1)\ T(\mathbf{e}_2)\ T(\mathbf{e}_3)\ \ldots\ T(\mathbf{e}_n)]$. In this section, we want to determine a matrix C so that $T([\mathbf{v}]_{\mathcal{B}}) = C[\mathbf{v}]_{\mathcal{B}}$. Necessarily, C will be based on the basis $\mathcal{B}$ as well as the matrix A.

In what follows, we are going to consider the basis

$$\mathcal{B} = \left\{ \mathbf{b}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 2 \end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix} 1 \\ 1 \\ -2 \\ 1 \end{bmatrix}, \mathbf{b}_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 3 \end{bmatrix}, \mathbf{b}_4 = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 2 \end{bmatrix} \right\},$$

and the linear operator $T : \mathbb{R}^4 \to \mathbb{R}^4$ where $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\right) = \begin{bmatrix} x_1 - x_2 + x_3 + 2x_4 \\ 2x_1 - 3x_4 \\ x_1 + x_2 + x_3 \\ -3x_3 + x_4 \end{bmatrix}$.

1. Determine the standard matrix A for T.

2. Find $T(\mathbf{v})$ for the vector $\mathbf{v} = \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \end{bmatrix}$ using the definition of T and using the standard matrix A.

3. Now that we have $T(\mathbf{v})$, we can easily determine $[T(\mathbf{v})]_{\mathcal{B}}$. Calculate this explicitly and also determine an expression for $[T(\mathbf{v})]_{\mathcal{B}}$ in terms of the matrices A and B and the vector $\mathbf{v}$. Verify that your solution works by determining scalars c_1, c_2, c_3, c_4 so that $T(\mathbf{v}) = c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + c_3\mathbf{b}_3 + c_4\mathbf{b}_4$.

4. Next, we are going to build the matrix C so that $T([\mathbf{v}]_{\mathcal{B}}) = C[\mathbf{v}]_{\mathcal{B}}$. Then we will check to see if this equation holds for the vector $\mathbf{v}$ in the previous question.

   (a) Determine $[T(\mathbf{b}_1)\ T(\mathbf{b}_2)\ T(\mathbf{b}_3)\ T(\mathbf{b}_4)]$ both explicitly and in terms of the matrix A and the vectors $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, \mathbf{b}_4$.

   (b) The matrix $[T(\mathbf{b}_1)\ T(\mathbf{b}_2)\ T(\mathbf{b}_3)\ T(\mathbf{b}_4)]$ is the product of what two matrices?

   (c) How do we find the $\mathcal{B}$-coordinate vector for each $T(\mathbf{b}_i)$? That is, this can be accomplished by multiplication by what matrix?

   (d) Find $C = [[T(\mathbf{b}_1)]_{\mathcal{B}}\ [T(\mathbf{b}_2)]_{\mathcal{B}}\ [T(\mathbf{b}_3)]_{\mathcal{B}}\ [T(\mathbf{b}_4)]_{\mathcal{B}}]$ explicitly and in terms of the matrices A and B and the vectors $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, \mathbf{b}_4$.

   (e) We denote the matrix in the last question by $[T]_{\mathcal{B}}$ and it is called the **matrix representation of T with respect to $\mathcal{B}$**. It is the product of what three matrices?

5. Now let's check to see if this new matrix works. Determine $[\mathbf{v}]_{\mathcal{B}}$ for the vector $\mathbf{v}$ in Question 2 and then $[T]_{\mathcal{B}}[\mathbf{v}]_{\mathcal{B}}$. Did you get the same vector as in Question 3? That is, is it true that $[T]_{\mathcal{B}}[\mathbf{v}]_{\mathcal{B}} = [T(\mathbf{v})]_{\mathcal{B}}$?

6. To summarize, if $T : \mathbb{R}^n \to \mathbb{R}^n$ is linear operator with standard matrix A, then the **matrix representation of T with respect to basis $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$** for $\mathbb{R}^n$ is

$$[T]_{\mathcal{B}} = \underline{\hspace{3cm}}$$

where $B =$ _______________.

7. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be linear operator with standard matrix A. Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$ be a basis for $\mathbb{R}^n$. Let $B = [\mathbf{b}_1\ \mathbf{b}_2\ \ldots\ \mathbf{b}_n]$. Let $\mathbf{v}$ be a vector in $\mathbb{R}^n$. Fill in each of the following:

   (a) $[\mathbf{v}]_{\mathcal{B}} =$ _______________.

   (b) $T(\mathbf{v}) =$ _______________.

   (c) $[T(\mathbf{v})]_{\mathcal{B}} =$ _______________.

   (d) $[T]_{\mathcal{B}} =$ _______________.

   (e) $[T]_{\mathcal{B}}[\mathbf{v}]_{\mathcal{B}} =$ ______________________________.

8. Does you work in the preceding question verify that $[T(\mathbf{v})]_{\mathcal{B}} = [T]_{\mathcal{B}}[\mathbf{v}]_{\mathcal{B}}$? Why or why not?

9. Determine $[T]_{\mathcal{B}}$ if $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_2 \\ x_2 - x_3 \\ x_1 + 2x_4 \\ x_2 - x_3 + 3x_4 \end{bmatrix}$ and $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ -2 \end{bmatrix} \right\}$.

Use $[T]_{\mathcal{B}}$ to find $[T(\mathbf{v})]_{\mathcal{B}}$ for the vector $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ -1 \end{bmatrix}$.