

Section 2.8: One-to-one Linear Transformations

Today we are going to learn about **one-to-one** linear transformations.

Definition: A linear transformation $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is **one-to-one** if $\mathbf{u} \neq \mathbf{v}$ in \mathbb{R}^n , then $T(\mathbf{u}) \neq T(\mathbf{v})$ in \mathbb{R}^m , or, taking the contrapositive, if $T(\mathbf{u}) = T(\mathbf{v})$ in \mathbb{R}^m , then $\mathbf{u} = \mathbf{v}$ in \mathbb{R}^n .

We have already noted that for every linear transformation $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$, there exists an $m \times n$ matrix A such that $T(\mathbf{x}) = A\mathbf{x}$. Let's see what this says about the matrix A and how this relates to previously learned concepts (hint: Section 1.7).

Suppose $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a linear transformation.

1. Let \mathbf{u} and \mathbf{v} be vectors in \mathbb{R}^n such that $T(\mathbf{u}) = T(\mathbf{v})$. Show that $T(\mathbf{u} - \mathbf{v}) = \mathbf{0}$.

2. Based on your work in the previous question, we see that whenever $T(\mathbf{u}) = T(\mathbf{v})$, then there exists a vector \mathbf{w} , namely $\mathbf{w} = \mathbf{u} - \mathbf{v}$, such that $T(\mathbf{w}) = \mathbf{0}$. Thus, if we want to decide if T is one-to-one, it makes sense to look at the set of solutions to the equation $T(\mathbf{x}) = \mathbf{0}$.
 - (a) We have previously seen that the equation $T(\mathbf{x}) = \mathbf{0}$ has at least one solution. What is it?

 - (b) If $T(\mathbf{w}) = \mathbf{0}$ for some **nonzero** vector \mathbf{w} in \mathbb{R}^n , is T one-to-one? Why or why not?

 - (c) If T is one-to-one, how many solutions are there to the equation $T(\mathbf{x}) = \mathbf{0}$?

 - (d) If A is the standard matrix for T and T is one-to-one, how many solutions are there to the equation $A\mathbf{x} = \mathbf{0}$?

Since solving the equation $T(\mathbf{x}) = \mathbf{0}$ is integral to determining whether or not T is one-to-one, we make the following definition:

Definition: The **null space** of T is the set of solutions to the equation $T(\mathbf{x}) = \mathbf{0}$, that is, null space $T = \{\mathbf{x} \in \mathbb{R}^n \mid T(\mathbf{x}) = \mathbf{0}\}$.

Note: finding the null space of a linear transformation T with standard matrix A is the same as finding the general solution to the equation $A\mathbf{x} = \mathbf{0}$.

3. Using your work in the previous question, we have discovered that if $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a one-to-one linear transformation with $m \times n$ standard matrix A , then the only solution to $A\mathbf{x} = \mathbf{0}$ is $\mathbf{x} = \mathbf{0}$. Since you have seen this statement before, complete the following:

A linear transformation $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ with standard matrix A is one-to-one if and only if

- (a) null space $T =$ _____
- (b) The only solution to $A\mathbf{x} = \mathbf{0}$ is _____.
- (c) rank $A =$ _____
- (d) nullity $A =$ _____
- (e) Columns of A are _____
- (f) There is a pivot position in every _____ of A .
- (g) $A\mathbf{x} = \mathbf{b}$ has at most _____ solution for every $\mathbf{b} \in \mathbb{R}^m$.
- (h) The columns of $\text{rref}(A)$ are distinct _____

4. Consider the linear transformation $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ defined by $T(\mathbf{x}) = \mathbf{T} \left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right) = \begin{bmatrix} 5x_1 - 3x_2 \\ -7x_1 + 8x_2 \\ 2x_1 \end{bmatrix}$.

- (a) Find the standard matrix A for T .

- (b) Find the null space of T . Is T one-to-one? Why or why not?

5. Consider the linear transformation $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ defined by $T(\mathbf{x}) = \mathbf{T} \left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \right) = \begin{bmatrix} 3x_1 - 4x_2 \\ 2x_1 + x_3 \end{bmatrix}$.

(a) Find the standard matrix A for T .

(b) Find the null space of T . Is T one-to-one? Why or why not?

6. Suppose $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a linear transformation. Mark each of the following **True or False** and explain.

(a) If $n > m$, then T is not one-to-one.

(b) If $n \leq m$, then T is one-to-one.

(c) If T is one-to-one and onto, then $m = n$.